

HOMOLOGY SMALE-BARDEN MANIFOLDS WITH K-CONTACT AND SASAKIAN STRUCTURES

VICENTE MUÑOZ, JUAN ANGEL ROJO, AND ALEKSY TRALLE

ABSTRACT. Kollár has found subtle obstructions to the existence of Sasakian structures on 5-dimensional manifolds. In the present article we develop methods of using these obstructions to distinguish K-contact manifolds from Sasakian ones. In particular, we find the first example of a closed 5-manifold M with $H_1(M) = 0$ which is K-contact but which carries no semi-regular Sasakian structures.

1. INTRODUCTION

Sasakian geometry has become an important and active subject, especially after the appearance of the fundamental treatise of Boyer and Galicki [4]. Chapter 7 of this book contains an extended discussion of the topological problems in the theory of Sasakian, and, more generally, K-contact manifolds. These are odd-dimensional analogues to Kähler and symplectic manifolds, respectively.

The precise definition is as follows. Let (M, η) be a co-oriented contact manifold with a contact form $\eta \in \Omega^1(M)$, that is $\eta \wedge (d\eta)^n > 0$ everywhere, with $\dim M = 2n + 1$. We say that (M, η) is *K-contact* if there is an endomorphism Φ of TM such that:

- $\Phi^2 = -\text{Id} + \xi \otimes \eta$, where ξ is the Reeb vector field of η (that is $i_\xi \eta = 1$, $i_\xi(d\eta) = 0$),
- the contact form η is compatible with Φ in the sense that $d\eta(\Phi X, \Phi Y) = d\eta(X, Y)$, for all vector fields X, Y ,
- $d\eta(\Phi X, X) > 0$ for all nonzero $X \in \ker \eta$, and
- the Reeb field ξ is Killing with respect to the Riemannian metric defined by the formula $g(X, Y) = d\eta(\Phi X, Y) + \eta(X)\eta(Y)$.

In other words, the endomorphism Φ defines a complex structure on $\mathcal{D} = \ker \eta$ compatible with $d\eta$, hence Φ is orthogonal with respect to the metric $g|_{\mathcal{D}}$. By definition, the Reeb vector field ξ is orthogonal to $\ker \eta$, and it is a Killing vector field.

Let (M, η, g, Φ) be a K-contact manifold. Consider the contact cone as the Riemannian manifold $C(M) = (M \times \mathbb{R}^{>0}, t^2 g + dt^2)$. One defines the almost complex structure I on $C(M)$ by:

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- $I(X) = \Phi(X)$ on $\ker \eta$,
- $I(\xi) = t \frac{\partial}{\partial t}$, $I(t \frac{\partial}{\partial t}) = -\xi$, for the Killing vector field ξ of η .

We say that (M, η, Φ, g, I) is *Sasakian* if I is integrable. Thus, by definition, any Sasakian manifold is K-contact.

There is much interest on constructing K-contact manifolds which do not admit Sasakian structures. The odd Betti numbers up to the middle dimension of Sasakian manifolds must be even. The parity of b_1 was used to produce the first examples of K-contact manifolds with no Sasakian structure [4, example 7.4.16]. More refined tools are needed in the case of even Betti numbers. The cohomology algebra of a Sasakian manifold satisfies a hard Lefschetz property [6]. Using it examples of K-contact non-Sasakian manifolds are produced in [7] in dimensions 5 and 7. These examples are nilmanifolds with even Betti numbers, so in particular they are not simply connected.

The fundamental group can also be used to construct K-contact non-Sasakian manifolds. Fundamental groups of Sasakian manifolds are called Sasaki groups, and satisfy strong restrictions. Using this it is possible to construct (non-simply connected) compact manifolds which are K-contact but not Sasakian [10]. Also it has been used to provide an example of a solvmanifold of dimension 5 which satisfies the hard Lefschetz property and which is K-contact and not Sasakian [8].

When one moves to the case of simply connected manifolds, K-contact non-Sasakian examples of any dimension ≥ 9 were constructed in [14] using the evenness of the third Betti number of a compact Sasakian manifold. Alternatively, using the hard Lefschetz property for Sasakian manifolds there are examples [19] of simply connected K-contact non-Sasakian manifolds of any dimension ≥ 9 .

In [22] and in [2] the rational homotopy type of Sasakian manifolds is studied. In [2] it is proved that all higher order Massey products for simply connected Sasakian manifolds vanish, although there are Sasakian manifolds with non-vanishing triple Massey products. This yields examples of simply connected K-contact non-Sasakian manifolds in dimensions ≥ 17 . However, Massey products are not suitable for the analysis of lower dimensional manifolds.

The problem of the existence of simply connected K-contact non-Sasakian compact manifolds (open problem 7.4.1 in [4]) is still open in dimension 5. It was solved for dimensions ≥ 9 in [6, 7, 14] and for dimension 7 in [20] by a combination of various techniques based on the homotopy theory and symplectic geometry. In the least possible dimension the problem appears to be much more difficult. Here one has to use the arguments of [17] which give subtle obstructions associated to the classification of Kähler surfaces. By definition, a simply connected compact oriented 5-manifold is called a *Smale-Barden manifold*. These manifolds are classified topologically by $H_2(M, \mathbb{Z})$ and the second Stiefel-Whitney class. Chapter 10 of the book by Boyer and Galicki is devoted to a description of some Smale-Barden manifolds which carry Sasakian structures. The following problem is still open (open problem 10.2.1 in [4]).

Do there exist Smale-Barden manifolds which carry K-contact but do not carry Sasakian structures?

In the present paper we make the first step towards a positive answer for the above question. A *homology Smale-Barden manifold* is a compact 5-dimensional manifold with $H_1(M, \mathbb{Z}) = 0$. A Sasakian structure is regular if the leaves of the Reeb flow are a foliation by circles with the structure of a circle bundle over a smooth manifold. The Sasakian structure is quasi-regular if the foliation is a Seifert circle bundle over a (cyclic) orbifold. It is semi-regular if this foliation has only locus of non-trivial isotropy of codimension 2, that is, if the base orbifold is a topological manifold. Any manifold admitting a Sasakian structure has also a quasi-regular Sasakian structure. Semi-regularity is only a small extra requirement. With this notions, our main result is:

Theorem 1. *There exists a homology Smale-Barden manifold which admits a semi-regular K-contact structure but which does not carry any semi-regular Sasakian structure.*

In order to put our result into a general context, it is worth recalling Kollár's obstructions to Sasakian structures [17]. If a 5-dimensional manifold M has a Sasakian structure, then it has a quasi-regular Sasakian structure. Then it is a Seifert bundle structure over a Kähler orbifold with isotropy locus a collection of complex curves. The second homology $H_2(M, \mathbb{Z})$ allows to recover the genus of these curves and if they are disjoint. In [17], 5-manifolds M are constructed which are Seifert bundles over 4-orbifolds X with isotropy being surfaces not satisfying the adjunction equality, hence X cannot be Kähler and M cannot be Sasakian.

To produce K-contact 5-dimensional manifolds we need to produce symplectic 4-dimensional orbifolds with suitable symplectic surfaces. Such K-contact 5-manifold cannot admit a Sasakian structure if we prove that such configuration of surfaces (genus and disjointness condition) cannot be produced for a Kähler orbifold with complex curves. We propose the following conjecture:

There does not exist a Kähler manifold or a Kähler orbifold with $b_1 = 0$ and $b_2 \geq 2$ having b_2 disjoint complex curves all of genus $g \geq 1$.

We give the first result in this direction (Theorem 29). Our construction of a K-contact 5-manifold which does not admit a Sasakian structure relies on producing a symplectic 4-manifold with b_2 disjoint symplectic surfaces of genus $g \geq 1$, but also with genus $g \leq 3$, to fit with our needs in Theorem 29. This is the content of the delicate construction in Section 5.

To the awareness of the authors, this is the first construction of a K-contact manifold which is quasi-regular and non-regular. All previous explicit constructions of K-contact manifolds are given as circle bundles over smooth symplectic manifolds, and hence they are regular.

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2. 4-DIMENSIONAL CYCLIC ORBIFOLDS

An n -dimensional (differentiable) orbifold is a space endowed with an atlas $\{(\tilde{U}_\alpha, \phi_\alpha, \Gamma_\alpha)\}$, where $\tilde{U}_\alpha \subset \mathbb{R}^n$, $\Gamma_\alpha < \text{GL}(\mathbb{R}^n)$ is a finite group acting linearly, and $\phi_\alpha : \tilde{U}_\alpha \rightarrow U_\alpha \subset X$ is a Γ_α -invariant map which induces a homeomorphism $\tilde{U}_\alpha/\Gamma_\alpha \cong U_\alpha$ onto an open set U_α of X . There is also a condition of compatibility of charts: for each point $p \in U_\alpha \cap U_\beta$ there is some $U_\gamma \subset U_\alpha \cap U_\beta$ with $p \in U_\gamma$, monomorphisms $\iota_{\gamma\alpha} : \Gamma_\gamma \hookrightarrow \Gamma_\alpha$, $\iota_{\gamma\beta} : \Gamma_\gamma \hookrightarrow \Gamma_\beta$, and open embeddings $f_{\gamma\alpha} : \tilde{U}_\gamma \rightarrow \tilde{U}_\alpha$, $f_{\gamma\beta} : \tilde{U}_\gamma \rightarrow \tilde{U}_\beta$, which satisfy $\iota_{\gamma\alpha}(g)(f_{\gamma\alpha}(x)) = f_{\gamma\alpha}(g(x))$ and $\iota_{\gamma\beta}(g)(f_{\gamma\beta}(x)) = f_{\gamma\beta}(g(x))$, for $g \in \Gamma_\gamma$.

As the groups Γ_α are finite, we can arrange (after a suitable conjugation) that $\Gamma_\alpha < \text{O}(n)$. The orbifold is *orientable* if all $\Gamma_\alpha < \text{SO}(n)$ and the embeddings $f_{\gamma\alpha}$ preserve orientation. Note that for any point $x \in X$, we can arrange always a chart $\phi : \tilde{U} \rightarrow U$ with $\tilde{U} \subset \mathbb{R}^n$ is a ball centered at 0 and $\phi(0) = x$, and $\tilde{U}/\Gamma \cong U$, with $\Gamma < \text{SO}(n)$. In this case, we call Γ the *isotropy group* at x . A *cyclic* orbifold has all isotropy groups which are cyclic groups $\Gamma \cong \mathbb{Z}_m$, and $m = m(x)$ is the order of the isotropy at x . In this paper, all we shall work exclusively with 4-dimensional cyclic oriented orbifolds, which we shall address just as orbifolds.

Let X be such an orbifold. Take $x \in X$ and a chart $\phi : \tilde{U} \rightarrow U$ around x . Let $\Gamma = \mathbb{Z}_m < \text{SO}(4)$ be the isotropy group. Then U is homeomorphic to an open neighbourhood of $0 \in \mathbb{R}^4/\mathbb{Z}_m$. A matrix of finite order in $\text{SO}(4)$ is conjugate to a diagonal matrix in $\text{U}(2)$ of the type $(\exp(2\pi i j_1/m), \exp(2\pi i j_2/m)) = (\xi^{j_1}, \xi^{j_2})$, where $\xi = e^{2\pi i/m}$. Therefore we can suppose that $\tilde{U} \subset \mathbb{C}^2$ and $\Gamma = \mathbb{Z}_m = \langle \xi \rangle \subset \text{U}(2)$ acts on \tilde{U} as

$$\xi \cdot (z_1, z_2) := (\xi^{j_1} z_1, \xi^{j_2} z_2). \quad (1)$$

Here j_1, j_2 are defined modulo m . As the action is effective, we have $\gcd(j_1, j_2, m) = 1$. Let us list the possible local models for an action given by the formula (1).

We call $x \in X$ a regular point if $m(x) = 1$, otherwise we call it a (non-trivial) isotropy point. We say that $D \subset X$ is an isotropy surface of multiplicity m if D is closed, and there is a dense open subset $D^\circ \subset D$ which is a surface and $m(x) = m$, for $x \in D^\circ$. From the topological point of view, we call $x \in X$ a smooth point if a neighbourhood of x is homeomorphic to a ball in \mathbb{R}^4 , and singular otherwise. Clearly a regular point is smooth, but not conversely as we shall see next.

Proposition 2. *Let X be a (cyclic, oriented, 4-dimensional) orbifold and $x \in X$ with local model $\mathbb{C}^2/\mathbb{Z}_m$. Then there are at most two isotropy surfaces D_i , with multiplicity $m_i|m$, through x . If there are two such surfaces D_i, D_j , then they intersect transversely and $\gcd(m_i, m_j) = 1$. The fundamental group of the link of x has order d with $(\prod m_i)d = m$, the product over all m_i such that $x \in D_i$. So the point is smooth if and only if $\prod m_i = m$.*

Proof. For an action given by (1), we set $m_1 := \gcd(j_1, m)$, $m_2 := \gcd(j_2, m)$. Note that $\gcd(m_1, m_2) = 1$, so we can write $m_1 m_2 d = m$, for some integer d . Put

$j_1 = m_1 e_1$, $j_2 = m_2 e_2$, $m = m_1 c_1 = m_2 c_2$. Clearly $c_1 = m_2 d$ and $c_2 = m_1 d$ and $d = \gcd(c_1, c_2)$.

We have five cases:

- (a) x is an isolated singular point. This corresponds to $m_1 = m_2 = 1$. As $\gcd(j_1, m) = \gcd(j_2, m) = 1$, the only fixed point is $(0, 0)$ since any power of ξ rotates both copies of \mathbb{C} non trivially. In this case the quotient space is singular, and the singularity is a cone over a lens space S^3/\mathbb{Z}_m , which is the link of the origin. Note that $d = m$.
- (b) Two isotropy surfaces and x is a smooth point, $m_1, m_2 > 1$, $d = 1$. Let us see that the action is equivalent to the product of one action on each factor \mathbb{C} . In this case $c_2 = m_1$ and $c_1 = m_2$. So $\gcd(c_1, c_2) = 1$ and $m = c_1 c_2$. The action is given by $\xi \cdot (z_1, z_2) := (\exp(2\pi i e_1/c_1) z_1, \exp(2\pi i e_2/c_2) z_2)$. We see that

$$\begin{aligned}\xi^{c_1} \cdot (z_1, z_2) &= (z_1, \exp(2\pi i c_1 e_2/c_2) z_2), \\ \xi^{c_2} \cdot (z_1, z_2) &= (\exp(2\pi i c_2 e_1/c_1) z_1, z_2),\end{aligned}$$

so the surfaces $D_1 = \{(z_1, 0)\}$ and $D_2 = \{(0, z_2)\}$ have isotropy groups $\langle \xi^{c_1} \rangle = \mathbb{Z}_{m_1}$ and $\langle \xi^{c_2} \rangle = \mathbb{Z}_{m_2}$, respectively. In this case $m = m_1 m_2$, $d = 1$.

Note that $\mathbb{Z}_m = \langle \xi^{c_1} \rangle \times \langle \xi^{c_2} \rangle$ if and only if $d = \gcd(c_1, c_2) = 1$. In this case the action of \mathbb{Z}_m decomposes as the product of the actions of \mathbb{Z}_{m_2} and \mathbb{Z}_{m_1} on each of the factors \mathbb{C} . The quotient space is $\mathbb{C}^2/\mathbb{Z}_m \cong \mathbb{C}/\mathbb{Z}_{m_2} \times \mathbb{C}/\mathbb{Z}_{m_1}$, which is homeomorphic to $\mathbb{C} \times \mathbb{C}$, and hence x is a smooth point (its link is S^3).

- (c) Two isotropy surfaces intersect at x and x is a singular point. In this case $d = \gcd(c_1, c_2) > 1$ and $m_1, m_2 > 1$. Now $\langle \xi^{c_1}, \xi^{c_2} \rangle = \langle \xi^d \rangle = \mathbb{Z}_{m'}$ with $d m' = m$. As $m' = m_1 m_2$, case (b) applies to the action of ξ^d and the quotient space is $\mathbb{C}^2/\mathbb{Z}_{m'} \cong \mathbb{C}/\mathbb{Z}_{m_2} \times \mathbb{C}/\mathbb{Z}_{m_1}$, which is homeomorphic to a ball in \mathbb{C}^2 via the map $(z_1, z_2) \mapsto (w_1, w_2) = (z_1^{m_2}, z_2^{m_1})$. The points of $D_1 = \{(w_1, 0)\}$ and $D_2 = \{(0, w_2)\}$ define two surfaces intersecting transversely, and with multiplicities m_1, m_2 , respectively.

Now ξ acts on $\mathbb{C}^2/\mathbb{Z}_{m'}$ by the formula $\xi \cdot (w_1, w_2) = (\xi^{m_2 j_1} w_1, \xi^{m_1 j_2} w_2) = (\exp(2\pi i e_1/d) w_1, \exp(2\pi i e_2/d) w_2)$, where $\gcd(e_1, d) = \gcd(e_2, d) = 1$. Therefore this action falls into case (a). The quotient is therefore $\mathbb{C}^2/\langle \xi \rangle \cong (\mathbb{C}/\mathbb{Z}_{m_2} \times \mathbb{C}/\mathbb{Z}_{m_1})/\mathbb{Z}_d$, the point x has as link a lens space S^3/\mathbb{Z}_d , and the images of D_1 and D_2 are the points with non-trivial isotropy, with isotropies m_1, m_2 , respectively.

- (d) One isotropy surface and x is a smooth point. In this case $m_2 = 1$ and $m_1 = m$. As $d = 1$, this is basically as case (b). The action is $\xi \cdot (z_1, z_2) = (z_1, \exp(2\pi i j_2/m) z_2)$. There is only one surface $D_1 = \{(z_1, 0)\}$ with non-trivial isotropy m , and all its points have the same isotropy. The quotient $\mathbb{C}^2/\mathbb{Z}_m = \mathbb{C} \times (\mathbb{C}/\mathbb{Z}_m)$ is topologically smooth.
- (e) One isotropy surface and x is a singular point. In this case $m_2 = 1$, $m_1 d = m$ and $d > 1$. This is basically as case (c). Now $c_2 = m$ and $c_1 = d$. Let $d m' = m$ so $m' = m_1$. The quotient space $\mathbb{C}^2/\mathbb{Z}_{m'} \cong \mathbb{C} \times \mathbb{C}/\mathbb{Z}_{m_1}$ is homeomorphic to a ball in \mathbb{C}^2 and the points of $D_1 = \{(z_1, 0)\}$ define a

surface with isotropy m_1 . Now for the quotient $\mathbb{C}^2/\mathbb{Z}_m = (\mathbb{C} \times (\mathbb{C}/\mathbb{Z}_{m_1}))/\mathbb{Z}_d$, the image of D_1 consists of points with isotropy m_1 , except for the origin which has isotropy $m = m_1d$. The link around x is the lens space S^3/\mathbb{Z}_d , hence it is singular. The rest of the points of D_1 are smooth.

□

Definition 3. *We say that an orbifold X is smooth if all its points are smooth. That is, all points of X fall into cases (b) or (d) in Proposition 2. This is equivalent to X being a topological manifold.*

Note that in case (b), we can change the generator $\xi = e^{2\pi i/m}$ of \mathbb{Z}_m to $\xi' = \xi^k$ for k such that $ke_i \equiv 1 \pmod{m_i}$, $i = 1, 2$, so that $\xi' \cdot (z_1, z_2) = (\exp(\frac{2\pi i}{m_2})z_1, \exp(\frac{2\pi i}{m_1})z_2)$. With this new generator, the action has model \mathbb{C}^2 with the action $\xi \cdot (z_1, z_2) = (\xi^{m_1}z_1, \xi^{m_2}z_2)$, $\xi = e^{2\pi i/m}$. Similar remark applies to case (d).

Note that according to Definition 3, a smooth orbifold is not a smooth manifold. However, there is a mechanism to produce a smooth orbifold from a smooth manifold. This is a standard result, but we include the proof since we have not found it in the literature.

Proposition 4. *Let X be a smooth (oriented) 4-manifold with embedded surfaces D_i intersecting transversely, and coefficients $m_i > 1$ such that $\gcd(m_i, m_j) = 1$ if D_i, D_j intersect, then there is a smooth orbifold X with isotropy surfaces D_i of multiplicities m_i .*

Proof. We consider X with its atlas as smooth manifold. We start by fixing a Riemannian metric such that in a neighbourhood of the (finitely many) points which are in the intersection of two of the D_i 's, it is standard, that is, for $x \in D_i \cap D_j$ there is a chart $f : B_\epsilon(0) \times B_\epsilon(0) \subset \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow U$, with $f(0,0) = x$, $D_i \cap U = f(B_\epsilon(0) \times \{0\})$, $D_j \cap U = f(\{0\} \times B_\epsilon(0))$, and g is the standard metric on U .

Now let $x \in X$ be a point. If x does not lie in any D_i , take a smooth chart $f : B_\epsilon(0) \subset \mathbb{R}^4 \rightarrow U$ to a neighbourhood U of x not touching any D_i . Then we consider the orbifold chart $(B_\epsilon(0), f, \{1\})$.

If x lies in only one $D = D_i$ with $m = m_i$, take a chart as follows. Take a small neighbourhood $V \subset D$ of x , and by using coordinates we identify $V \subset \mathbb{R}^2$. Consider the exponential map from the normal bundle (on V) N_D to X , $\exp : N_D \rightarrow X$. For small $\epsilon > 0$, $\exp : N_D^\epsilon = \{(x, v) | x \in V, v \in (T_x D)^\perp, |v| < \epsilon\} \rightarrow X$ is a diffeomorphism onto its image. Trivialize the normal bundle, so that $N_D^\epsilon \cong V \times B_\epsilon(0)$. This gives a smooth chart $f : V \times B_\epsilon(0) \rightarrow U$, $f(w_1, w_2) = \exp_{w_1}(w_2)$, with coordinates (w_1, w_2) . We define the following orbifold chart: consider $\tilde{U} = V \times B_\epsilon(0)$ and $\phi : \tilde{U} = V \times B_\epsilon(0) \rightarrow U$, by $\phi(z_1, z_2) = f(z_1, re^{2\pi i m \theta})$, for $z_2 = re^{2\pi i \theta}$. The action of \mathbb{Z}_m is given by $\xi \cdot (z_1, z_2) = (z_1, \xi z_2)$, $\xi = e^{2\pi i/m}$. This defines a chart $(\tilde{U}, \phi, \mathbb{Z}_m)$ at x .

If x lies in the intersection of two surfaces, say D_1, D_2 , with coefficients m_1, m_2 , then $\gcd(m_1, m_2) = 1$, by assumption. Take small neighbourhoods $V_1 \subset D_1$,

$V_2 \subset D_2$ of x , which we identify with balls $B_\epsilon(0) \subset \mathbb{R}^2$. Consider a smooth chart $f : B_\epsilon(0) \times B_\epsilon(0) \subset \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow U$, with $f(0,0) = x$, $D_1 \cap U = f(B_\epsilon(0) \times \{0\})$, $D_2 \cap U = f(\{0\} \times B_\epsilon(0))$, and g is the standard metric on U . We define the orbifold chart as follows: consider $\tilde{U} = B_\epsilon(0) \times B_\epsilon(0)$ and $\phi : \tilde{U} \rightarrow U$, $\phi(z_1, z_2) = \phi(r_1 e^{2\pi i \theta_1}, r_2 e^{2\pi i \theta_2}) = f(r_1 e^{2\pi i m_2 \theta_1}, r_2 e^{2\pi i m_1 \theta_2})$. The action of $\mathbb{Z}_m = \mathbb{Z}_{m_2} \times \mathbb{Z}_{m_1}$, $m = m_1 m_2$, is given by $\xi \cdot (z_1, z_2) = (\xi^{m_1} z_1, \xi^{m_2} z_2)$, where $\xi = e^{2\pi i/m}$. Then $(\tilde{U}, \phi, \mathbb{Z}_m)$ is a chart at x .

It is easy to see that these charts are compatible with the \mathbb{Z}_m -actions. We only need to check this near a surface D , where the change of charts are those of the respective normal bundle N_D . So, if the changes of charts for the smooth structure of X near the D are of the form $(w_1, w_2) \mapsto (w'_1, w'_2) = (\varphi(w_1), h(w_1)w_2)$, for some smooth maps $\varphi : V_\alpha \subset \mathbb{C} \rightarrow V_\beta \subset \mathbb{C}$, $h : V_\alpha \rightarrow S^1$, then the changes of charts for the orbifold structure of X are of the form $(z_1, z_2) \mapsto (z'_1, z'_2) = (\varphi(z_1), h(z_1)^{1/m} z_2)$, for some m -th root of h , and so they are smooth and \mathbb{Z}_m -equivariant. \square

Note that we have not introduced the freedom of choosing coefficients j_i for each D_i . We claim that one can always arrange so that the local actions are as above. First take a neighborhood that intersects just one isotropy surface D_i . As $\gcd(j_i, m_i) = 1$, changing the generator of \mathbb{Z}_{m_i} , we can arrange that the action of \mathbb{Z}_{m_i} has $j_i = 1$, so it is given by $\xi \cdot (z_1, z_2) = (z_1, \xi z_2)$. Finally, whenever D_i, D_j intersect, as $\gcd(m_i, m_j) = 1$, we can arrange simultaneously $j_i = m_i, j_j = m_j$ at x (by (b) of Proposition 2). This means that using a different set of j_i does not change the resulting orbifold.

Also a smooth (cyclic, oriented) 4-orbifold X can be converted into a smooth manifold with the same underlying space such that the isotropy surfaces are embedded submanifolds intersecting transversely. As we shall not use this construction, we do not include the proof.

Let X be an orbifold with atlas $\{(\tilde{U}_\alpha, \phi_\alpha, \Gamma_\alpha)\}$. An orbi-tensor on X is a collection of tensors T_α on each \tilde{U}_α which are Γ_α -equivariant, and which agree under the changes of charts. In particular, we have orbi-differential forms $\Omega_{orb}^p(X)$, orbi-Riemannian metrics g , and orbi-almost complex structures J . The exterior differential, covariant derivatives, Lie bracket, Nijenhuis tensor, etc, are defined in the usual fashion.

Definition 5. A symplectic orbifold (X, ω) is an orbifold X with a $\omega \in \Omega_{orb}^2(X)$ such that $d\omega = 0$ and $\omega^n > 0$, where $2n = \dim X$.

An almost Kähler orbifold (X, J, ω) consists of an orbifold X , and orbi-almost complex structure J and an orbi-symplectic form ω such that $g(u, v) = \omega(u, Jv)$ defines an orbi-Riemannian metric.

A Kähler orbifold is an almost Kähler orbifold satisfying the integrability condition that the Nijenhuis tensor $N_J = 0$. This is equivalent to requiring that the changes of charts are biholomorphisms of open sets of \mathbb{C}^n .

The following (presumably well-known) result will be useful in the following. It allows to have a nice local picture of the intersection of two symplectic surfaces in a symplectic 4-manifold.

Lemma 6. *Let (X, ω) be a symplectic 4-manifold, and suppose that $S, N \subset X$ are symplectic surfaces intersecting transversely and positively. Then we can perturb S (small in the C^0 -sense) so that S is symplectic, S and N intersect ω -orthogonally, and the perturbation only changes S near the points of intersection with N .*

Moreover, once we perturb S , there are Darboux coordinates (z, w) near all the intersection points of N and S in which $N = \{z = 0\}$ and $S = \{w = 0\}$.

Proof. We can arrange that the intersection becomes orthogonal after a small symplectic isotopy around the intersection point. Suppose we are working in a Darboux chart (z, w) with symplectic form $\omega = -\frac{i}{2}(dz \wedge d\bar{z} + dw \wedge d\bar{w})$, let N be given by the equation $N = \{z = 0\}$, and S be given as the graph of a map $w = az + b\bar{z} + g(z)$, where $a, b \in \mathbb{C}$, $|g(z)| \leq C|z|^2$, $|\partial_z g(z)| + |\partial_{\bar{z}} g(z)| \leq C|z|$. The condition for D to be symplectic and intersect positively N is that $|a|^2 - |b|^2 + 1 > 0$.

One can deform S locally to

$$S' = \{(z, \rho(|z|/\epsilon)^{2\alpha}(az + b\bar{z} + g(z)))\},$$

for some $\epsilon > 0$ and $\alpha > 0$ to be determined later, where $\rho(t)$ is a bump function which is 0 on $[0, 1]$ and 1 on $[2, \infty)$. Clearly S' intersects N at $(0, 0)$ orthogonally with respect to ω . An easy calculation gives that

$$\begin{aligned} \omega|_{S'} &= -\frac{i}{2}(dz \wedge d\bar{z} + dw \wedge d\bar{w}) \\ &= \left(1 + \left(2\alpha\rho\rho' \cdot \left(\frac{|z|}{\epsilon}\right)^{2\alpha} + \rho^2\right)(|a|^2 - |b|^2) + O\left(\frac{|z|^{2\alpha+1}}{\epsilon^{2\alpha}} + |z|\right)\right) \frac{-idz \wedge d\bar{z}}{2}, \end{aligned} \quad (2)$$

where $\rho = \rho(|z|/\epsilon)^{2\alpha}$, $\rho' = \rho'(|z|/\epsilon)^{2\alpha}$. Now

$$2\alpha\rho\rho' \cdot \left(\frac{|z|}{\epsilon}\right)^{2\alpha} \leq \alpha C 2^{2\alpha+1} < \delta,$$

for any small $\delta > 0$, by choosing α small enough. If $|a|^2 - |b|^2 \geq 0$, clearly

$$1 + \left(2\alpha\rho\rho' \cdot \left(\frac{|z|}{\epsilon}\right)^{2\alpha} + \rho^2\right)(|a|^2 - |b|^2) > 0.$$

If $0 > |a|^2 - |b|^2 > -1$, then

$$1 + \left(2\alpha\rho\rho' \cdot \left(\frac{|z|}{\epsilon}\right)^{2\alpha} + \rho^2\right)(|a|^2 - |b|^2) > 1 + (\delta + 1)(|a|^2 - |b|^2) > 0,$$

choosing $\delta > 0$ small enough. The error term in (2) is $O(\epsilon)$, so it can be neglected for ϵ small enough. This completes the proof that S' is a symplectic surface. Clearly, close to $(0, 0)$, S' is given by the equation $w = 0$. \square

For constructing symplectic orbifolds, we have the following result. Again, this seems to be fairly well-known, but we have not been able to find a proof in the literature.

Proposition 7. *Let X be a symplectic smooth (oriented) 4-manifold with symplectic surfaces D_i intersecting transversely and positively, and coefficients $m_i > 1$ such that $\gcd(m_i, m_j) = 1$ if D_i, D_j intersect. Then there is a smooth symplectic orbifold X with isotropy surfaces D_i of multiplicities m_i .*

Proof. By Lemma 6, we can assume that the surfaces D_i intersect orthogonally. As in the proof of Proposition 4, we start by fixing a metric. We do this as follows. First at each point at an intersection $D_i \cap D_j$, fix a Darboux chart $f : B_\epsilon(0) \times B_\epsilon(0) \rightarrow U$ with $D_i \cap U = f(B_\epsilon(0) \times \{0\})$ and $D_j \cap U = f(\{0\} \times B_\epsilon(0))$. Take a standard metric on U , and the corresponding almost complex structure J_U on U . Fix now compatible almost complex structures J_i on each D_i (that is, $J_i : T_x D_i \rightarrow T_x D_i$ at each $x \in D_i$), which agree on U with J_U . The normal bundle N_{D_i} over D_i is a symplectic bundle. Take a Riemannian metric on N_{D_i} compatible with its symplectic structure, and define a Riemannian metric on each $T_x X = T_x D_i \oplus N_{D_i, x}$, $x \in D_i$, by declaring the direct sum orthogonal. We extend this metric g on $\bigcup D_i$ to a Riemannian metric on the whole of X compatible with the symplectic form. This produces an almost Kähler structure on the whole of X for which each D_i is a J -invariant surface.

Now we use this metric g for producing the atlas of Proposition 4 that gives X the structure of a smooth orbifold. Let us now construct the orbifold symplectic form. We need first to modify ω to a nearby ω' as follows.

Let $D = D_i$ be one of the isotropy surfaces. On N_D^ϵ we have a radial coordinate r , and an angular coordinate θ , well-defined in every chart up to addition of a function on D . By construction, we have $\omega = \omega|_D + r dr \wedge d\theta$ along D . For the bundle $N_D \rightarrow D$, consider a connection 1-form $\eta \in \Omega^1(N_D - D)$, and let $F = d\eta \in \Omega^2(N_D)$ be its curvature. Thus $\Omega = r dr \wedge \eta - \frac{1}{2} r^2 F + \omega|_D$, is a closed form on N_D that coincides with ω along D . In the last expression, $\omega|_D$ stands for the pull-back of $\omega|_D$ by the bundle projection. Now $|\Omega - \omega| \leq Cr$, where C is a constant independent of r . On N_D^ϵ , $\Omega - \omega$ is closed so (being zero on D) it is exact, say $\Omega - \omega = d\beta$.

We can choose the 1-form β so that it satisfies $|\beta| \leq Cr^2$, by the usual standard procedure to produce a primitive of an exact form. Indeed, if $\Omega - \omega = \alpha_0 \wedge dr + \alpha_1$, one takes $\beta = \int_0^r \alpha_0 dr$ (see [3, p. 34]). This β is continuous, and smooth outside $\{r = 0\}$.

We also arrange the 1-form η to be equal to $d\theta$ on $U \cap N_D^\epsilon$, so that $F = 0$ on $U \cap N_D^\epsilon$ and so $\Omega = \omega$ on $U \cap N_D^\epsilon$. These forms Ω 's for the different D 's paste to a globally defined Ω on a neighbourhood of $\bigcup D_i$.

Take a cut-off function $\rho : [0, \epsilon] \rightarrow [0, 1]$ with $\rho(r) \equiv 1$ for $r \in [0, \frac{1}{3}\epsilon]$, and $\rho(r) \equiv 0$ for $r \in [\frac{2}{3}\epsilon, \epsilon]$, and $|\rho'| \leq C/\epsilon$. Hence $\omega' = \omega + d(\rho\beta)$ satisfies that it is equal to Ω for

$|r| \leq \frac{1}{3}\epsilon$, equal to ω for $|r| \geq \frac{2}{3}\epsilon$, and $|\omega' - \omega| = |d(\rho\beta)| = |d\rho \wedge \beta + \rho \wedge d\beta| \leq C\epsilon$. This produces a globally defined 2-form ω' on X . For ϵ small enough, ω' is symplectic.

Now let us define our orbi-symplectic form. Take first a point x in some $D = D_i$ and not in U . We have smooth coordinates (w_1, w_2) , $w_2 = re^{2\pi i\theta}$, and orbifold coordinates (z_1, z_2) , $z_1 = w_1$ and $z_2 = re^{2\pi i\vartheta}$, $\theta = m\vartheta$. The action is $\xi \cdot (z_1, z_2) = (z_1, \xi z_2)$. Here $\omega' = \Omega = \alpha + r dr \wedge d\theta + r dr \wedge \gamma$, where α is a 2-form and γ is a 1-form, and both α and γ are invariant in the fiber direction, in particular $\text{SO}(2)$ -equivariant (recall that the connection 1-form is $\eta = d\theta + \gamma$).

We set, in the orbifold coordinates (z_1, r, ϑ) ,

$$\hat{\omega} = \alpha + m r dr \wedge d\vartheta + r dr \wedge \gamma.$$

This is closed, smooth, symplectic and \mathbb{Z}_m -invariant. Moreover, $\hat{\omega}$ agrees with the pull-back of ω' via the orbifold chart $(z_1, z_2) \mapsto (w_1, w_2)$, and this implies that $\hat{\omega}$ is invariant by the orbifold change of charts.

Finally, on U , we take smooth coordinates (w_1, w_2) , $w_1 = r_1 e^{2\pi i\theta_1}$, $w_2 = r_2 e^{2\pi i\theta_2}$, and orbifold coordinates are $z_1 = r_1 e^{2\pi i\vartheta_1}$, $z_2 = r_2 e^{2\pi i\vartheta_2}$, with $\theta_1 = m_2\vartheta_1$, $\theta_2 = m_1\vartheta_2$. Here $\omega' = r_1 dr_1 \wedge d\theta_1 + r_2 dr_2 \wedge d\theta_2$. We set

$$\hat{\omega} = m_2 r_1 dr_1 \wedge d\vartheta_1 + m_1 r_2 dr_2 \wedge d\vartheta_2,$$

which defines an orbifold symplectic form on U . \square

Remark 8. We observe that in the proof of Proposition 7, $[\omega'] = [\omega]$. This is checked by integrating along any oriented surface $S \subset X$. Take S to intersect transversely all D_i . Let S_δ to be S less small δ -balls around the intersections $S \cap D_i$. Then $\langle [\omega'] - [\omega], [S] \rangle = \int_S d(\rho\beta) = \lim \int_{S_\delta} d(\rho\beta) = \lim \int_{\partial S_\delta} \rho\beta = \lim \int_{\partial S_\delta} \beta = 0$, since $|\beta| \leq Cr^2$. Note that β is not C^∞ at $r = 0$, that is the reason for this indirect argument.

Consider the orbifold forms $(\Omega_{\text{orb}}(X), d)$. Their cohomology is denoted $H_{\text{orb}}^*(X)$. This is isomorphic to the usual De Rham cohomology [9, p. 8], $H_{\text{orb}}^*(X) \cong H_{DR}^*(X)$. The isomorphism can be explicitly constructed as follows: take a smooth map $\varphi : X \rightarrow X$ such that it is the identity off a neighbourhood of $\bigcup D_i$, and contracts radially a smaller neighbourhood of each D_i to D_i , followed by a map that contracts a neighbourhood of each point in an intersection $D_i \cap D_j$ to the point. Then the map $\varphi^* : \Omega_{\text{orb}}^*(X) \rightarrow \Omega^*(X)$ gives the isomorphism $\varphi^* : H_{\text{orb}}^*(X) \rightarrow H_{DR}^*(X)$.

The orbifold form $\hat{\omega}$ defines a class in $H_{\text{orb}}^2(X)$ and this is $[\hat{\omega}] = [\omega']$ under the above isomorphism.

Lemma 9. Let (X, ω) be a symplectic orbifold. Then (X, ω) admits the structure of an almost Kähler orbifold.

Proof. We have to adapt the usual construction of an almost Kähler structure for a symplectic manifold. This can be found in [5, p. 68]. Choose an orbifold metric g' , and define the orbifold $(1, 1)$ -tensor A by $g'(AX, Y) = \omega(X, Y)$. Then AA^* is positive definite and symmetric and hence it is a well-defined square root $\sqrt{AA^*}$, which is an orbifold section of $\text{End}_{\text{orb}}(TM)$. Then $J = (\sqrt{AA^*})^{-1}A$ is an orbifold

$(1, 1)$ -tensor (i.e. $J \in \text{End}_{\text{orb}}(TM)$) and it clearly satisfies $J^2 = -\text{id}$. Recall that $\omega(JX, JY) = \omega(X, Y)$, and $\omega(X, JX) = g'(\sqrt{AA^*}X, X)$, so the orbi-metric g is compatible with (ω, J) is $g(X, Y) = g'(\sqrt{AA^*}X, Y)$.

Since this local construction of J and g is canonical (once we have chosen the orbi-metric g'), it is compatible with change of charts, so this defines an orbifold almost Kähler structure. \square

Proposition 10. *If X is a smooth Kähler cyclic orbifold, then X is a smooth complex manifold and D_i are complex curves intersecting transversely.*

Proof. As the almost-complex structure is integrable, we can take the orbifolds charts $\phi : \tilde{U} \rightarrow U \subset X$ to be holomorphic, with $\tilde{U} \subset \mathbb{C}^2$. The group $\Gamma = \mathbb{Z}_m$ acts by a biholomorphism $f : \tilde{U} \rightarrow \tilde{U}$. The map $\varphi(z) = \frac{1}{m} \sum_{k=0}^{m-1} f^k(d_0 f^{-k}(z))$ defines a new chart $\phi' = \phi \circ \varphi$, where the action of Γ is linear, since $\varphi(d_0 f(z)) = f(\varphi(z))$.

So $\Gamma < \text{GL}(2, \mathbb{C})$ acts by complex transformations, and the quotient \tilde{U}/Γ has a natural complex structure (that is, the complex structure on the complement of $\bigcup D_i$ extends naturally to $\bigcup D_i$). The induced map $\bar{\phi} : \tilde{U}/\Gamma \rightarrow U$ is holomorphic, and thus biholomorphic since it is bijective. These maps define an atlas as a complex manifold. Note that if (z_1, z_2) are the coordinates for an orbifold holomorphic chart, with action $\xi \cdot (z_1, z_2) = (\xi^{m_2} z_1, \xi^{m_1} z_2)$, $\xi = e^{2\pi i/m}$, $m = m_1 m_2$, then $w_1 = z_1^{m_1}$, $w_2 = z_2^{m_2}$ define holomorphic coordinates for the quotient. The surfaces D_i are defined by the equations $w_1 = 0$ or $w_2 = 0$ in such charts, therefore they are smooth complex curves intersecting transversely. \square

3. SEIFERT BUNDLES

A Seifert bundle is a space fibered by circles over an orbifold. We give a precise definition.

Definition 11. *Let X be a cyclic, oriented n -dimensional orbifold. A Seifert bundle over X is an oriented $(n+1)$ -dimensional manifold M equipped with a smooth S^1 -action and a continuous map $\pi : M \rightarrow X$ such that for an orbifold chart $(\tilde{U}, \phi, \mathbb{Z}_m)$, there is a commutative diagram*

$$\begin{array}{ccc} (S^1 \times \tilde{U})/\mathbb{Z}_m & \xrightarrow{\cong} & \pi^{-1}(U) \\ \downarrow \pi & & \downarrow \pi \\ \tilde{U}/\mathbb{Z}_m & \xrightarrow{\cong} & U \end{array}$$

where the action of \mathbb{Z}_m on S^1 is by multiplication by $\xi = e^{2\pi i/m}$ and the top diffeomorphism is S^1 -equivariant.

Proposition 12. *An oriented $(n+1)$ -manifold endowed with a fixed point free action of S^1 is a Seifert bundle over a cyclic, oriented n -orbifold.*

Proof. Let M be a manifold endowed with a fixed point free action of S^1 . Then X will be the space of leaves of the S^1 -action. The orbifold structure on X is obtained

as follows. Take an accessory Riemannian metric g and average it over S^1 to make it S^1 -invariant. For a point $p \in M$, let $O(p)$ be the orbit of p . Let $I(p) = \mathbb{Z}_m = \langle \xi \rangle$, $\xi = e^{2\pi i/m}$, be the isotropy of p . Then the action of ξ , say $f : M \rightarrow M$, fixes p and the tangent direction R_p to the orbit $O(p)$. Hence the differential of $d_p f$ fixes the orthogonal hyperplane $H_p = R_p^\perp$, inducing an action of \mathbb{Z}_m on it. Since M is oriented, $d_p f$ preserves orientation, so $\mathbb{Z}_m = \langle d_p f \rangle < \text{SO}(n)$.

For a small $\tilde{U} \subset H_p$, the exponential map and the S^1 -action give a local diffeomorphism $\varphi : S^1 \times \tilde{U} \rightarrow M$, $\varphi(u, z) = u \cdot \exp_p(z)$. On $S^1 \times \{0\}$ the isotropy is \mathbb{Z}_m , hence a neighbourhood of $O(p) = \varphi(S^1 \times \{0\})$ is modelled on $(S^1 \times \tilde{U})/\mathbb{Z}_m$. This action is by multiplication by ξ on the S^1 -factor, and by the action of $d_p f$ on \tilde{U} . The space of leaves is identified with \tilde{U}/\mathbb{Z}_m , and $(\tilde{U}, \bar{\varphi}, \mathbb{Z}_m)$ gives the desired orbifold chart, $\bar{\varphi} : \tilde{U} \rightarrow \tilde{U}/\mathbb{Z}_m \subset X$. \square

Suppose in the following that X is a 4-dimensional orbifold and $\pi : M \rightarrow X$ is a Seifert bundle over X . According to the normal form of the \mathbb{Z}_m -action given in (1), the open subset $\pi^{-1}(U) \cong (S^1 \times \tilde{U})/\mathbb{Z}_m$ is parametrized by $(u, z_1, z_2) \in S^1 \times \mathbb{C}^2$, modulo the \mathbb{Z}_m -action $\xi \cdot (u, z_1, z_2) = (\xi u, \xi^{j_1} z_1, \xi^{j_2} z_2)$, for some integers j_1, j_2 , where $\xi = e^{2\pi i/m}$. The S^1 -action is given by $s \cdot (u, z_1, z_2) = (su, z_1, z_2)$, so $\mathbb{Z}_m \subset S^1$ is the isotropy group of $O(p) \subset M$, and the exponents j_1, j_2 are determined by the S^1 -action.

We say that $\{(D_i, m_i, j_i)\}$ are the *orbit invariants* of the Seifert bundle if $D_i \subset X$ are the isotropy surfaces, with multiplicities m_i , and the local model around a point $p \in D_i^\circ = D_i - \bigcup_{i \neq j} (D_i \cap D_j)$ is of the form $(S^1 \times \tilde{U})/\mathbb{Z}_{m_i}$ with action $\xi \cdot (u, z_1, z_2) = (\xi u, z_1, \xi^{j_i} z_2)$, $D_i = \{z_2 = 0\}$. If the orbifold is smooth, then for a point $p \in D_i \cap D_j$, the local model is of the form $(S^1 \times \tilde{U})/\mathbb{Z}_{m_i}$ with action $\xi \cdot (u, z_1, z_2) = (\xi u, \xi^{j_j} z_1, \xi^{j_i} z_2)$, $D_i = \{z_2 = 0\}$, $D_j = \{z_1 = 0\}$.

Definition 13. For a Seifert bundle $\pi : M \rightarrow X$, we define its Chern class as follows. Let $\mu = \mathbb{Z}_{m(X)}$, where $m(X) = \text{lcm}\{m(x) \mid x \in X\}$. Consider the circle fiber bundle $M/\mu \rightarrow X$ and its Chern class $c_1(M/\mu) \in H^2(X, \mathbb{Z})$. We define

$$c_1(M/X) = \frac{1}{m(X)} c_1(M/\mu) \in H^2(X, \mathbb{Q}).$$

The next proposition shows that the orbit invariants determine the Seifert bundle globally when X is smooth.

Proposition 14. Let X be an oriented 4-manifold and $D_i \subset X$ oriented surfaces of X which intersect transversely. Let $m_i > 1$ such that $\text{gcd}(m_i, m_j) = 1$ if D_i and D_j intersect. Let $0 < j_i < m_i$ with $\text{gcd}(j_i, m_i) = 1$ for every i . Let $0 < b_i < m_i$ such that $j_i b_i \equiv 1 \pmod{m_i}$. Finally, let B be a complex line bundle on X . Then there is a Seifert bundle $f : M \rightarrow X$ with orbit invariants $\{(D_i, m_i, j_i)\}$ and first Chern class

$$c_1(M/X) = c_1(B) + \sum_i \frac{b_i}{m_i} [D_i]. \quad (3)$$

The set of all such Seifert bundles forms a principal homogeneous space under $H^2(X, \mathbb{Z})$, where the action corresponds to changing B .

Proof. The orbit invariants determine uniquely the structure of smooth orbifold for X . Let $\{(\tilde{U}_\alpha, \phi_\alpha, \Gamma_\alpha)\}$ be a covering of X by orbifold charts. The orbit invariants determine the local model of the Seifert bundle, so the possible Seifert bundles with these orbifold invariants are given by the gluing of the local models. This is defined by transition functions $g_{\alpha\beta} : \tilde{U}_\alpha \cap \tilde{U}_\beta \rightarrow S^1$ which are Γ_γ -invariant for $\tilde{U}_\gamma \subset \tilde{U}_\alpha \cap \tilde{U}_\beta$. Therefore it is defined by a 1-cocycle in $\mathcal{C}_{orb}^\infty(S^1)$, the orbifold functions with values in S^1 . Using the exponential short exact sequence of sheaves $0 \rightarrow \mathbb{Z} \rightarrow \mathcal{C}_{orb}^\infty \rightarrow \mathcal{C}_{orb}^\infty(S^1) \rightarrow 0$, where the sheaf of orbifold real functions \mathcal{C}_{orb}^∞ is a fine sheaf, we have that the possible Seifert bundles are parametrized by $H^1(X, \mathcal{C}_{orb}^\infty(S^1)) \cong H^2(X, \mathbb{Z})$. We can tensor $M \otimes B$, for a line bundle $B \rightarrow X$, by multiplying the transition functions. Therefore the set of Seibert bundles forms a homogeneous space under $H^2(X, \mathbb{Z})$.

For $M \otimes B$, we have that $(M \otimes B)/\mu = (M/\mu) \otimes B^{\otimes m}$, since the quotient is given locally by $(u, z_1, z_2) \mapsto (u^m, z_1, z_2)$, where $m = m(X)$. So

$$c_1((M \otimes B)/X) = \frac{1}{m} c_1((M \otimes B)/\mu) = \frac{1}{m} (c_1(M/\mu) + m c_1(B)) = c_1(M/X) + c_1(B).$$

To prove (3) is equivalent to prove that $c_1(M/\mu) = m c_1(M/X) \equiv \sum_i b_i \frac{m}{m_i} [D_i] \pmod{m}$. For this we take a 2-cycle $S \subset X$, that we can assume that it intersects transversely the D_j 's, and compute $\langle c_1(M/\mu), S \rangle$. To compute $c_1(M/\mu)$, we fix a transverse section s of the line bundle associated to M/μ .

In $\langle c_1(M/\mu), S \rangle$ there is a contribution coming from balls $B_p \subset S$ around each intersection point $p \in S \cap (\bigcup D_i)$ and a contribution from $S^o = S - \bigcup B_p$. The second one is $\langle c_1(M/\mu), S^o \rangle = \langle m c_1(M/X), S^o \rangle \in m\mathbb{Z}$, since $M \rightarrow X$ is an honest circle bundle over the locus S^o . For this equality we choose s to be the image of a section of the line bundle associated to the circle bundle $M \rightarrow S^o$.

Now we look at the circle bundle $M/\mu \rightarrow X$ at a point $p \in S \cap D_i$. We can arrange orbifold coordinates (z_1, z_2) such that $D_i = \{z_2 = 0\}$ and $S = \{z_1 = 0\}$. The Seifert bundle is given by coordinates (u, z_1, z_2) modulo $\xi \cdot (u, z_1, z_2) = (\xi u, z_1, \xi^{j_i} z_2)$, $\xi = e^{2\pi i/m_i}$. Equivalently, modulo $(u, z_1, z_2) \mapsto (\xi^{b_i} u, z_1, \xi z_2)$. The circle bundle M/μ is parametrized by $(v = u^m, z_1, z_2)$ modulo $(v, z_1, z_2) \mapsto (v, z_1, \xi z_2)$. The section s lifts to a section \hat{s} of M over ∂B_p . In orbifold coordinates of X , it is of the form $\hat{s}(z_1, z_2) = (u(z_1, z_2), z_1, z_2)$, with $u(z_1, \xi z_2) = \xi^{b_i} u(z_1, z_2)$. This means that we can choose $u(z_1, z_2) = z_2^{b_i}$. Therefore, the section s is locally $s(z_1, z_2) = (v, z_1, z_2)$ with $v = z_2^{b_i m}$. Going back to smooth coordinates $w_1 = z_1$, $w_2 = z_2^{m_i}$, the section is written as $s(w_1, w_2) = (w_2^{b_i m/m_i}, w_1, w_2)$. Therefore the zero set of s along $S = \{w_1 = 0\}$ has multiplicity $b_i m/m_i$. Adding the contributions of all points $p \in S \cap D_i$, we get the contribution $\sum \frac{b_i m}{m_i} \langle [D_i], S \rangle$ to $\langle c_1(M/\mu), S \rangle$. This proves the sought formula. \square

Let $\pi : M \rightarrow X$ be a Seifert bundle, $p \in M$ and $x = \pi(p)$. The fiber over x is the orbit $O(p)$, which is of the form S^1/\mathbb{Z}_m , where $m = m(x) = m(p)$ is both the isotropy of x (as orbifold point) and the isotropy of p (for the S^1 -action on M). We call the orbit $O(p)$ *semi-regular* if the orbifold point $x = \pi(p)$ is smooth. This means that the local model in Proposition 2 is of type (b) or (d). In the case (d), the orbit $O(p)$ has nearby orbits $O(p')$ with multiplicity $m(p') = m(p)$. In case (b), $m = m_1 m_2$ and $\gcd(m_1, m_2) = 1$, and $O(p)$ has nearby orbits $O(p_1)$ and $O(p_2)$ of multiplicities m_1, m_2 , respectively.

Definition 15. *We say that a Seifert bundle is semi-regular if the base orbifold X is smooth, that is all orbits are semi-regular.*

Now we want to relate the homology of M with that of X for a Seifert bundle $\pi : M \rightarrow X$. We only need the case of a semi-regular Seifert bundle, and we are interested in the case where $H_1(M, \mathbb{Z}) = 0$. We have the following result.

Theorem 16. *Suppose that $\pi : M \rightarrow X$ is a semi-regular Seifert bundle with isotropy surfaces D_i with multiplicities m_i . Then $H_1(M, \mathbb{Z}) = 0$ if and only if*

- (1) $H_1(X, \mathbb{Z}) = 0$,
- (2) $H^2(X, \mathbb{Z}) \rightarrow \sum H^2(D_i, \mathbb{Z}/m_i)$ is surjective,
- (3) $c_1(M/\mu) \in H^2(X, \mathbb{Z})$ is primitive.

Moreover, $H_2(M, \mathbb{Z}) = \mathbb{Z}^k \oplus \bigoplus (\mathbb{Z}/m_i)^{2g_i}$, $g_i = \text{genus of } D_i$, $k + 1 = b_2(X)$.

Proof. First, suppose that X is smooth and satisfies (1)–(3). We have $H^3(X, \mathbb{Z}) = H_1(X, \mathbb{Z}) = 0$, by Poincaré duality. By [17, Proposition 26(3)], we get $H_1(M, \mathbb{Z}) = 0$. Now [17, Corollary 27] gives that $b_2(M) = k$ and [17, Proposition 28] gives that $H_2(M, \mathbb{Z})_{\text{tors}} = \bigoplus (\mathbb{Z}/m_i)^{2g_i}$.

Conversely, if $H_1(M, \mathbb{Z}) = 0$ then the argument in [17, §25] gives that $H_1(X, \mathbb{Z}) = 0$. Then [17, Proposition 26(3)] implies conditions (2)–(3). \square

Corollary 17. *Suppose that M is a 5-manifold with $H_1(M, \mathbb{Z}) = 0$ and $H_2(M, \mathbb{Z}) = \mathbb{Z}^k \oplus \bigoplus_{i=1}^{k+1} (\mathbb{Z}/p^i)^{2g_i}$, $k \geq 0$, p a prime, and $g_i \geq 1$. If $M \rightarrow X$ is a semi-regular Seifert bundle, then $H_1(X, \mathbb{Z}) = 0$, $H_2(X, \mathbb{Z}) = \mathbb{Z}^{k+1}$, and the ramification locus has $k + 1$ disjoint surfaces D_i linearly independent in rational homology, and of genus $g(D_i) = g_i$.*

Proof. By Theorem 16, $H_1(X, \mathbb{Z}) = 0$ and $H_2(X, \mathbb{Z}) = \mathbb{Z}^{k+1}$. Let D_1, \dots, D_m be the isotropy surfaces. Their coefficients are numbers of the form p^i , $1 \leq i \leq k + 1$, hence it must be $m \geq k + 1$. Now $\mathbb{Z}^{k+1} \rightarrow \sum_{i=1}^m H^2(D_i, \mathbb{Z}/p^i)$ is surjective. In particular, it cannot be $m > k + 1$, so there are exactly $k + 1$ surfaces, and they have each a different isotropy coefficient p^i . This implies that D_i and D_j are disjoint for $i \neq j$.

The above map is given by

$$H^2(X; \mathbb{Z}) = \mathbb{Z}^{k+1} \longrightarrow \sum_{i=1}^{k+1} H^2(D_i, \mathbb{Z}/p^i) = \sum_{i=1}^{k+1} \mathbb{Z}/p^i$$

$$[S] \mapsto ([S] \cdot [D_i] \pmod{p^i})$$

As this map is surjective, for each $[D_i]$ there exists an element $[S_i] \in H^2(X; \mathbb{Z})$ so that $[S_i] \cdot [D_i] \equiv 1 \pmod{p^i}$ and $[S_i] \cdot [D_j] \equiv 0 \pmod{p^j}$ for $j \neq i$. Thus $[S_i] \cdot [D_i] \equiv 1 \pmod{p}$ and $[S_i] \cdot [D_j] \equiv 0 \pmod{p}$ for $j \neq i$. If the $[D_i]$ are not linearly independent then there exists integers b_i so that $\sum b_i [D_i] = 0$. We can choose b_i that are coprime. Multiplying by $[S_j]$ we get $\sum b_i [D_i] \cdot [S_j] = 0$, for $1 \leq j \leq k+1$. Reducing modulo p , we have $b_i \equiv 0 \pmod{p}$, which is a contradiction.

Finally it follows that $g(D_i) = g_i$. \square

4. K-CONTACT AND SASAKIAN 5-MANIFOLDS

A Sasakian or a K-contact structure on a compact manifold M is called *quasi-regular* if there is a positive integer δ satisfying the condition that each point of M has a neighbourhood such that each leaf for ξ passes through U at most δ times. If $\delta = 1$, then the Sasakian or K-contact structure is called *regular* (see [4, p. 188]).

A result of [21] says that if M admits a Sasakian structure, then it admits also a quasi-regular Sasakian structure. Also, if a compact manifold M admits a K-contact structure, it admits a quasi-regular contact structure [20].

Theorem 18. *Let (M, η, Φ, ξ, g) be a quasiregular K-contact manifold. Then the space of leaves X has a natural structure of an almost Kähler cyclic orbifold where the projection $M \rightarrow X$ is a Seifert bundle.*

Furthermore, if (M, η, Φ, ξ, g) is Sasakian, then X is a Kähler orbifold.

Proof. Take a point $p \in M$, and let $O(p)$ be the orbit through p . Since $O(p)$ intersects finitely many times every small neighbourhood, then $O(p)$ must be a circle. Let ϕ_t be the Reeb flow, and consider t_p the period of $\phi_t(p)$. Let $f = \phi_{t_p}$, $H_p = \langle \xi_p \rangle^\perp = \ker \eta|_p$, and $d_p f : H_p \rightarrow H_p$. For $\epsilon > 0$ small, take $B_\epsilon(0) \subset H_p$. Then $\varphi : \mathbb{R} \times B_\epsilon(0) \rightarrow X$, $\varphi(t, w) = \phi_t(\exp_p(w))$, is an open embedding whose image W is a neighbourhood of $O(p)$ consisting of orbits of the Reeb flow (recall that the Reeb flow is by isometries, so it preserves the distances to $O(p)$). Since ξ is a quasi-regular vector field, the orbits intersect $S_p = \varphi(\{0\} \times B_\epsilon(0))$ at finitely many points. For $q = \varphi(0, w)$, the points of intersection are $f^k(q) = \varphi(k t_p, w)$, $k \in \mathbb{Z}$. So there is some k such that $f^k(q) = q$, i.e., $d_p f^k(w) = w$. Therefore $d_p f : H_p \rightarrow H_p$ is of finite order. Let m be its order. So $d_p f^m = \text{Id}$, hence $f^m = \text{Id}$. Therefore ϕ_t gives an S^1 -action with period $m t_p$.

By Proposition 12, we have a Seifert bundle $\pi : M \rightarrow X$, over the space of leaves X , which is a cyclic orbifold. Let us see that X has the structure of an almost Kähler orbifold. The open set $W \cong (S^1 \times B_\epsilon(0))/\mathbb{Z}_m$, and the orbifold

chart is $(\tilde{U} = B_\epsilon(0), \phi, \mathbb{Z}_m)$, where $\phi : B_\epsilon(0) \rightarrow X$, $\phi(w) = \pi(\varphi(0, w))$. Then the orbifold tangent space at p is identified with $T_0\tilde{U} \cong H_p$. We put the complex and symplectic structures J, ω on $T_0\tilde{U}$ given by $\Phi, d\eta$ on H_p , respectively. These are well defined independently of the point in the orbit, since the Reeb flow acts by isometries, preserving Φ and η . Finally, these complex and symplectic structures are \mathbb{Z}_m -invariant (since the action is given by $d_p f$, the isometry defined by the Reeb flow $f = \phi_{t_p}$).

Now suppose that M is Sasakian. Then, by definition, there is an integrable complex structure I on the cone $C(M) = M \times \mathbb{R}^{>0}$, given by $I(X) = \Phi(X)$ on $\ker \eta$, and $I(\xi) = t \frac{\partial}{\partial t}$. This means that the Nijenhuis tensor $N_I(X, Y) = -[X, Y] + I[IX, Y] + I[X, IY] - [IX, IY] = 0$. Take an orbifold chart $(\tilde{U}, \phi, \mathbb{Z}_m)$ as above with $W = (S^1 \times \tilde{U})/\mathbb{Z}_m$. Take X, Y two \mathbb{Z}_m -equivariant vector fields on \tilde{U} . These define vector fields X', Y' on $W \times \mathbb{R}^{>0} = ((S^1 \times \tilde{U})/\mathbb{Z}_m) \times \mathbb{R}^{>0}$, where $X'_p, Y'_p \in H_p$, for all points p . So $-[X', Y'] + \Phi[\Phi X', Y'] + \Phi[X', \Phi Y'] - [\Phi X', \Phi Y'] = 0$. Now lift to $S^1 \times \tilde{U} \times \mathbb{R}^{>0}$ and project down to \tilde{U} . The Lie bracket is preserved, $\pi_*[X', Y'] = [X, Y]$, and $\pi_*(\Phi X') = JX$. So we get $N_J(X, Y) = -[X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] = 0$, as required.

□

Lemma 19. *Let (X, ω) be a symplectic 4-manifold with a collection of embedded symplectic surfaces D_i intersecting transversally and positively, and integer numbers $m_i > 1$, with $\gcd(m_i, m_j) = 1$ whenever $D_i \cap D_j \neq \emptyset$. Then there is a Seifert bundle $\pi : M \rightarrow X$ such that:*

- (1) *It has Chern class $c_1(M/X) = [\hat{\omega}]$ for some orbifold symplectic form $\hat{\omega}$ on X .*
- (2) *If $\sum \frac{b_i m}{m_i} [D_i]$ is primitive and the second Betti number $b_2(X) \geq 3$, then then we can further have that $c_1(M/\mu) \in H^2(X, \mathbb{Z})$ is primitive.*

Proof. Consider the Seifert bundle $\pi : M \rightarrow X$ given by some orbit invariants $\{(D_i, m_i, j_i)\}$, $m = m(X)$, with $c_1(M/\mu) = m \sum \frac{b_i}{m_i} [D_i]$, possible by Proposition 14. The set of elements

$$\left\{ \frac{1}{mk+1}a + \frac{1}{mk+1}c_1(M/X) \mid a \in H^2(X, \mathbb{Z}), k \geq 1 \right\} \subset H^2(X, \mathbb{R}) \quad (4)$$

is dense. So we can perturb ω slightly so that $[\omega] = \frac{1}{mk+1}a + \frac{1}{mk+1}c_1(M/X)$, for some $a \in H^2(X, \mathbb{Z})$ and $k \geq 1$. Then the symplectic form $\tilde{\omega} = (mk+1)\omega$ satisfies that $[\tilde{\omega}] = a + c_1(M/X)$. Choosing a line bundle B with $c_1(B) = a$, we have a Seifert bundle $\tilde{M} = M \otimes B$ with $c_1(\tilde{M}/X) = [\tilde{\omega}]$. Now the process of Proposition 7 gives an orbifold symplectic form $\hat{\omega}$ on the orbifold X with isotropy surfaces D_i with multiplicities m_i . This has $[\hat{\omega}] = [\tilde{\omega}] = c_1(\tilde{M}/X)$. This proves (1).

Now let us see (2). Take a primitive class $b_1 \in H^2(X, \mathbb{Z})$ with $c_1(\tilde{M}/\mu) \cdot b_1 = 0$. Then there exists $a_0 \in H^2(X, \mathbb{Z})$ with $a_0 \cdot b_1 = 1$. Now take a primitive $b_2 \in H^2(X, \mathbb{Z})$ with $c_1(\tilde{M}/\mu) \cdot b_2 = 0$ and $a_0 \cdot b_2 = 0$, possible since $b_2(X) \geq 3$. Then

let us see that the elements of (4) with $\gcd(a \cdot b_1, a \cdot b_2) = 1$ are dense. Take any element x in (4) given by some a and $k \geq 1$. Let $k_1 = a \cdot b_1, k_2 = a \cdot b_2 \in \mathbb{Z}$. Consider k_0 a large integer containing all prime factors of both k_1, k_2 . Then take the element x' given by $a' = k_0 a + a_0, k' = k_0 k$, which satisfies $|x' - x| \leq C/k$. Thus the set of such x' is dense.

So consider an element a with $\gcd(a \cdot b_1, a \cdot b_2) = 1$ and a Seifert bundle with $c_1(\tilde{M}/\mu) = ma + c_1(M/\mu) = [\hat{\omega}]$ as above. Then $c_1(\tilde{M}/\mu) \cdot b_j = m(a \cdot b_j), j = 1, 2$. Therefore if $c_1(\tilde{M}/\mu)$ is divisible by some ℓ , then $\ell|m$. So $c_1(M/\mu) = c_1(\tilde{M}/\mu) - ma$ is divisible by ℓ , and hence it is not a primitive class, contrary to hypothesis. \square

Let $\pi : M \rightarrow X$ be any Seifert bundle. We construct a connection 1-form on $M \rightarrow X$ as follows. Take an orbifold covering $X = \bigcup U_\alpha$, with an orbifold partition of unity $\{\rho_\alpha\}$. For each $U_\alpha = \tilde{U}_\alpha/\mathbb{Z}_{m_\alpha}$ we have $\pi^{-1}(U_\alpha) = (S^1 \times \tilde{U}_\alpha)/\mathbb{Z}_{m_\alpha}$. Let $\eta_\alpha = u_\alpha^{-1} du_\alpha$, where u_α is the S^1 -coordinate. Define

$$\eta = \sum \rho_\alpha \eta_\alpha.$$

This is an orbifold 1-form and $F = d\eta = \sum d\rho_\alpha \wedge \eta_\alpha$ is the (orbifold) curvature 2-form of $M \rightarrow X$.

For the circle fiber bundle $M/\mu \rightarrow X$, η descends to a 1-form $\bar{\eta}$ on M/μ . The fiber of M/μ is parametrized by $\bar{u}_\alpha = u_\alpha^m, m = m(X)$. So the connection 1-form on M/μ equals $\bar{\eta} = m\eta$. Its curvature is mF and thus $c_1(M/\mu) = [mF]$. This implies that

$$c_1(M/X) = \frac{1}{m} c_1(M/\mu) = [F].$$

The following result appears in [4, p. 211], where it is referred to [15]. However the proof in [15] does not cover the orbifold case. So we have included a proof.

Theorem 20. *Let (X, ω, J, g) be an almost Kähler cyclic orbifold with $[\omega] \in H^2(X; \mathbb{Q})$, and let $\pi : M \rightarrow X$ be a Seifert bundle with $c_1(M/X) = [\omega]$. Then M admits a K-contact structure (ξ, η, Φ, g) such that $\pi^*(\omega) = d\eta$.*

Proof. Take the (orbifold) connection 1-form constructed above, and let $F = d\eta$ be its curvature. As $[F] = c_1(M/X) = [\omega]$, we have that $F - \omega = d\beta$, for some orbifold 1-form β . Then we can change η to $\eta' = \eta - \beta$, so that its curvature is $F' = F - d\beta = \omega$.

Now the 1-form η is a smooth form on the total space M . On each $\pi^{-1}(U) = (S^1 \times \tilde{U})/\mathbb{Z}_m$, we have that $d\eta = \omega$ is the 2-form coming from \tilde{U} . So $\eta \wedge (d\eta)^2 > 0$, and η is a contact form. Now define the Reeb vector field ξ as the one given by the S^1 -action, which clearly preserves η . Define $H_p = \ker \eta_p$, and $\Phi : T_p M \rightarrow T_p M$ by $\Phi(\xi) = 0$ and $\Phi : H_p \rightarrow H_p$ as the almost complex structure $J_x : T_x \tilde{U} \rightarrow T_x \tilde{U}$, for $x = \pi(p)$, under the isomorphism $H_p \cong T_x \tilde{U}$. This is well-defined since the S^1 -flow preserves the horizontal subspaces H_p . Clearly the Reeb flow preserves Φ .

Finally define the metric g by declaring H_p and ξ_p orthogonal, ξ_p unitary and g is the metric on H_p given by Φ and ω . Then the Reeb flow preserves g , i.e., it acts by isometries. This means that (M, ξ, η, Φ, g) is a K-contact manifold. \square

Theorem 20 corrects an statement of [16], where it is claimed that a K-contact structure can be constructed from an orbifold where the isotropy locus is not a symplectic surface.

5. A SYMPLECTIC 4-MANIFOLD WITH MANY DISJOINT SYMPLECTIC SURFACES

Now we move to the construction of a K-contact manifold which cannot admit a semi-regular Sasakian structure (Theorem 1). For this, we need a symplectic manifold with many disjoint symplectic surfaces which will be used to construct a Seifert bundle.

Theorem 21. *There exists a simply connected symplectic 4-manifold X with $b_2 = 36$ and with 36 disjoint surfaces S_1, \dots, S_{36} such that*

- (1) $g(S_1) = \dots = g(S_9) = 1$, $g(S_{11}) = \dots = g(S_{19}) = 1$, $g(S_{21}) = \dots = g(S_{29}) = 1$, and $S_i \cdot S_i = -1$, for $i = 1, \dots, 9, 11, \dots, 19, 21, \dots, 29$;
- (2) $g(S_{10}) = 3$, $g(S_{20}) = 3$, $g(S_{30}) = 3$, and $S_j \cdot S_j = -1$, $j = 10, 20, 30$;
- (3) $g(S_{31}) = 1$, $g(S_{32}) = 1$, $g(S_{33}) = 2$, and $S_{31} \cdot S_{31} = -1$, $S_{32} \cdot S_{32} = -1$, $S_{33} \cdot S_{33} = 1$;
- (4) $g(S_{34}) = 1$, $g(S_{35}) = 1$, $g(S_{36}) = 2$, and $S_{34} \cdot S_{34} = -1$, $S_{35} \cdot S_{35} = -1$, $S_{36} \cdot S_{36} = 1$.

The homology classes $[S_j]$, $j = 1, \dots, 36$, generate $H_2(X, \mathbb{Z})$.

In the subsequent subsections we will construct such X . Our basic tools are Gompf symplectic sum, symplectic blow-up, elliptic and Lefschetz fibrations, and symplectic resolution of transverse intersections. We recall these tools following [13].

5.1. Symplectic resolution of transverse intersections. Let X be a symplectic 4-manifold and let Σ_1 and Σ_2 be embedded symplectic surfaces intersecting transversely and positively at a point $q \in X$. Then $\Sigma_1 \cup \Sigma_2$ determines the homology class $[\Sigma_1] + [\Sigma_2] \in H_2(X, \mathbb{Z})$. By Lemma 6, after slightly perturbing Σ_1 we can take Darboux coordinates (z_1, z_2) in a 4-ball neighbourhood D of q , so that $\Sigma_1 = \{z_1 = 0\}$ and $\Sigma_2 = \{z_2 = 0\}$. Then the union $\Sigma_1 \cup \Sigma_2$ is described locally as

$$F = \{(z_1, z_2) \in D \mid z_1 z_2 = 0, |z_1|^2 + |z_2|^2 \leq 1\}.$$

Cut out the pair (D, F) and replace it with (D, R) , where $R \subset D$ is obtained by perturbing the subset

$$R' = \{(z_1, z_2) \mid z_1 z_2 = \varepsilon, |z_1|^2 + |z_2|^2 \leq 1\},$$

for $\varepsilon > 0$ sufficiently small, to achieve that $\partial F = \partial R \subset \partial D$. This construction replaces $\Sigma_1 \cup \Sigma_2$ by a smooth symplectic surface of genus $g(\Sigma_1) + g(\Sigma_2)$, representing

the homology class $[\Sigma_1] + [\Sigma_2]$. It does not change the ambient manifold. We call this construction the resolution of the transverse intersection.

5.2. Symplectic blow-up. Let X be a symplectic 4-manifold and $q \in X$. The symplectic blow-up of X at q is defined as follows. Take the Darboux coordinates (z_1, z_2) in a 4-ball neighbourhood D of q , and put the standard complex structure J on D . Consider

$$\tilde{D} = \{((z_1, z_2), [w_1, w_2]) \in D \times \mathbb{C}P^1 \mid z_1 w_2 = z_2 w_1\}.$$

Then there is a natural projection $q : \tilde{D} \rightarrow D$, such that $q : \tilde{D} - E \rightarrow D - \{(0, 0)\}$ is a biholomorphism, where $E = \{(0, 0)\} \times \mathbb{C}P^1$, $q(E) = \{(0, 0)\}$. We cut out D from X and replace it with \tilde{D} , obtaining the manifold \tilde{X} . The symplectic form of X and the natural symplectic form of \tilde{D} (coming from its Kähler structure) can be glued to give a symplectic structure for \tilde{X} . As a smooth 4-manifold, $\tilde{X} = X \# \overline{\mathbb{C}P^2}$ and $E = \overline{\mathbb{C}P^1} \subset \overline{\mathbb{C}P^2}$ is called the exceptional sphere. Its homology class $[E]$ is denoted by $e \in H_2(X', \mathbb{Z}) = H_2(X, \mathbb{Z}) \oplus H_2(\overline{\mathbb{C}P^2}, \mathbb{Z})$ and satisfies $e \cdot e = -1$.

Now consider a symplectic surface $\Sigma \subset X$ and blow up a point $p \in \Sigma$. Then we can take coordinates (z_1, z_2) such that $\Sigma = \{z_1 = 0\}$. The surface $\tilde{\Sigma} \subset \tilde{X}$ defined in \tilde{D} by the equations $z_1 = w_1 = 0$ is called the *proper transform* of Σ . It is symplectic, and $[\tilde{\Sigma}] = [\Sigma] - e$. Therefore $[\tilde{\Sigma}]^2 = [\Sigma]^2 - 1$. Moreover, the exceptional divisor E is symplectic and intersects $\tilde{\Sigma}$ transversely. Actually, the symplectic resolution of the intersection of $\tilde{\Sigma} \cup E$ is Σ .

If Σ_1 and Σ_2 are two symplectic surfaces in X intersecting transversely and positively at a point p , blowing-up at p and taking the proper transforms, we get two disjoint symplectic surfaces $\tilde{\Sigma}_1, \tilde{\Sigma}_2 \subset \tilde{X}$. This is proved by taking a Darboux chart such that $\Sigma_1 = \{z_1 = 0\}$ and $\Sigma_2 = \{z_2 = 0\}$, which is possible since Σ_1, Σ_2 intersect transversely and positively.

5.3. Gompf symplectic sum. The following construction is introduced in [12]. Let M_1 and M_2 be closed symplectic 4-manifolds, and $N_1 \subset M_1$, $N_2 \subset M_2$ symplectic surfaces of the same genus and with $N_1^2 = -N_2^2$. Fix a symplectomorphism $N_1 \cong N_2$. If ν_j is the normal bundle to N_j , then there is a reversing-orientation bundle isomorphism $\psi : \nu_1 \rightarrow \nu_2$. Identifying the normal bundles ν_i with the tubular neighbourhoods $\nu(N_j)$ of N_j in M_i , one has a symplectomorphism $\varphi : \nu(N_1) - N_1 \rightarrow \nu(N_2) - N_2$ by composing ψ with the diffeomorphism $x \mapsto \frac{x}{\|x\|^2}$ that turns each punctured normal fiber inside out. The Gompf symplectic sum $M_1 \#_N M_2$ is the manifold obtained from $(M_1 - N_1) \sqcup (M_2 - N_2)$ by gluing with φ above. It is proved in [12] that this surgery yields a symplectic manifold, denoted $M = M_1 \#_{N_1=N_2} M_2$. The Euler characteristic of the Gompf symplectic sum is given by $\chi(M) = \chi(M_1) + \chi(M_2) - 2\chi(N)$, where $N = N_1 = N_2$.

Lemma 22. *Suppose that $S_1 \subset M_1$ and $S_2 \subset M_2$ are symplectic surfaces intersecting transversely and positively with N_1, N_2 , respectively, such that $S_1 \cdot N_1 = S_2 \cdot N_2 = d$. Then S_1, S_2 can be glued to a symplectic surface $S = S_1 \# S_2 \subset M_1 \#_{N_1=N_2} M_2$ with self-intersection $S^2 = S_1^2 + S_2^2$ and genus $g(S) = g(S_1) + g(S_2) + d - 1$.*

Proof. When doing the Gompf symplectic sum of M_1, M_2 along N_1, N_2 , we arrange the symplectomorphism $N_1 \cong N_2$ to take the intersection points $S_1 \cap N_1$ to the points $S_2 \cap N_2$. Then we have to take tubular neighbourhoods of N_j by using the symplectic orthogonal to $T_p N_j$ at each $p \in N_j \cap S_j$. If S_j and N_j intersect orthogonally with respect to the symplectic form, then S_1 and S_2 glue nicely to give a symplectic surface S in the Gompf connected sum. We can arrange that the intersection becomes orthogonal after a small symplectic isotopy around the intersection point, as done in Lemma 6. The claim about the self intersection and the genus are straightforward. \square

5.4. Elliptic fibrations. We begin with some recollections on elliptic and Lefschetz fibrations from [12, 13]. A complex surface S is an elliptic fibration if there is a holomorphic map $f : S \rightarrow C$ to a complex curve C such that for generic $t \in C$ the preimages $f^{-1}(t)$ are smooth elliptic curves. The elliptic fibration $E(1)$ is defined on \mathbb{CP}^2 blown-up at 9 points as follows. Take two generic cubics in \mathbb{CP}^2 given by polynomials $p_0([x : y : z]) = 0$, $p_1([x : y : z]) = 0$. These cubics intersect in 9 points p_1, \dots, p_9 . Consider the pencil of cubics $t_0 p_0 + t_1 p_1$ parametrized by $[t_0 : t_1] \in \mathbb{CP}^1$. For any point $q \in \mathbb{CP}^2 - \{p_1, \dots, p_9\}$ there is only one cubic $t_0 p_0 + t_1 p_1$ going through q . This defines a map

$$f : \mathbb{CP}^2 - \{p_1, \dots, p_9\} \rightarrow \mathbb{CP}^1, \quad f(q) = [t_0 : t_1].$$

Blowing up \mathbb{CP}^2 at p_1, \dots, p_9 , we get a Kähler surface $E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$ and the map f extends to a $f : E(1) \rightarrow \mathbb{CP}^1$, which is an elliptic fibration.

We will use the notion of vanishing cycle. Let X be a Kähler manifold. A Lefschetz fibration on X is a holomorphic map $f : X \rightarrow \Sigma$, where Σ is a complex curve such that each critical point of f has a local (complex) coordinate chart on which $f(z_1, z_2) = z_1^2 + z_2^2$. Hence a regular fiber $F_t = f^{-1}(t)$ of the Lefschetz fibration is given locally by the equation $z_1^2 + z_2^2 = t$, and we can suppose $t > 0$ multiplying $(z_1, z_2) \in F_t$ by some complex number.

For $\epsilon > 0$ real and positive, the intersection $F_t \cap \mathbb{R}^2 \subset \mathbb{C}^2$ yields a circle $x_1^2 + x_2^2 = \epsilon$ (here $z_j = x_j + iy_j$). This circle bounds a disc D_ϵ in X defined by $\{(z_1, z_2) \in F_t \cap \mathbb{R}^2 \mid t \in [0, \epsilon]\} = X \cap \mathbb{R}^2 \cap B_\epsilon(0)$, which is called the vanishing cycle of the critical point. This is an embedded disc of self-intersection -1 and Lagrangian with respect to the symplectic structure of X . We refer to [13] for the detailed exposition of the theory of elliptic and Lefschetz fibrations.

Let us summarize the properties of $E(1)$ which will be used later, from [1, 13].

Proposition 23. *The elliptic fibration $E(1)$ has the following properties.*

- (1) $\pi_1(E(1)) = \{e\}$, $\chi(E(1)) = 12$, $b_2(E(1)) = 10$.
- (2) Every exceptional sphere E_i of the blow-up at a point p_i is a section of the elliptic fibration $f : E(1) \rightarrow \mathbb{CP}^1$, hence there are 9 disjoint sections.
- (3) Let $h \in H_2(\mathbb{CP}^2, \mathbb{Z})$ be the homology class of the line $L \subset \mathbb{CP}^2$, and e_i are homology classes of exceptional spheres E_i , then $H_2(E(1), \mathbb{Z}) = \langle h, e_1, \dots, e_9 \rangle$.

- (4) Let F be a generic fiber of $f : E(1) \rightarrow \mathbb{C}P^1$. Then $\pi_1(E(1) - F) = \{e\}$.
- (5) The homology class of F is $[F] = 3h - e_1 - \dots - e_9$. Hence a generic line $L \subset \mathbb{C}P^2$ intersects F transversely in 3 points.
- (6) For generic cubics, $f : E(1) \rightarrow \mathbb{C}P^1$ is a Lefschetz fibration.
- (7) In a fiber F there are 12 vanishing cycles; they come in two packets of six 1-cycles homologous to a and six 1-cycles homologous to b , where $\{a, b\}$ is a basis of $H_1(F, \mathbb{Z})$.

We have the following result.

Lemma 24. *Let X be a symplectic 4-manifold with an embedded symplectic surface $T \subset X$ of self-intersection zero and genus 1. Then the Gompf connected sum $X' = X \#_{T=F} E(1)$ has fundamental group $\pi_1(X') = \pi_1(X)/H$, where H is the normal subgroup generated by the image of $\pi_1(T) \rightarrow \pi_1(X)$.*

Proof. By definition $X' = (X - \nu(T)) \cup_B (E(1) - \nu(F))$, where $B = \partial(X - \nu(T)) = \partial(E(1) - \nu(F)) \cong \mathbb{T}^3$. Applying Seifert-Van Kampen theorem, $\pi_1(X')$ is isomorphic to the amalgamated product $\pi_1(X - \nu(T)) *_{\pi_1(B)} \pi_1(E(1) - \nu(F))$. Since $\pi_1(E(1) - \nu(F)) = \{1\}$, this is isomorphic to the quotient of $\pi_1(X - \nu(T))$ by the image of $\pi_1(B)$. Using Seifert-Van Kampen theorem for $X = (X - \nu(T)) \cup_B \nu(T)$, $\pi_1(X)$ is isomorphic to $\pi_1(X - \nu(T)) *_{\pi_1(B)} \pi_1(\nu(T))$. Therefore the quotient of $\pi_1(X)$ by the image of $\pi_1(T)$ equals the quotient of $\pi_1(X - \nu(T))$ by the image of $\pi_1(B)$. The result follows. \square

5.5. Making Lagrangian submanifolds symplectic. We will need a slight modification of Lemma 1.6 in [12].

Lemma 25. *Let (M, ω) be a 4-dimensional compact symplectic manifold. Assume that $[F_1], \dots, [F_k] \in H_2(M, \mathbb{Z})$ are linearly independent homology classes represented by k Lagrangian surfaces F_1, \dots, F_k which intersect transversely and not three of them intersect in a point. Then there is an arbitrarily small perturbation ω'' of the symplectic form ω such that all F_1, \dots, F_k become symplectic.*

Proof. Since $[F_1], \dots, [F_k]$ are linearly independent, there exists a closed 2-form η such that $\int_{F_i} \eta = 1$, for all $i = 1, \dots, k$. Take symplectic (volume) forms ω_i on F_i such that $\int_{F_i} \omega_i = 1$. Then $\int_{F_i} (\omega_i - j_i^* \eta) = 0$ so there are 1-forms α_i on F_i such that $\omega_i - j_i^* \eta = d\alpha_i$.

We extend α_i to a tubular neighbourhoods U_i of F_i by pulling-back via a projection $p_i : U_i \rightarrow F_i$. We arrange this projection to project any surface intersecting F_i to a point. Then we extend $p_i^* \alpha_i$ to the whole of M by multiplying with a cut-off function ρ_i which is 0 off a neighbourhood of F_i and 1 in a smaller neighbourhood. Set $\eta' = \eta + \sum_j d(\rho_j(p_j^* \alpha_j))$. Clearly, $d\eta' = d\eta = 0$ and $j_i^* \eta' = \omega_i$, for all i . The form $\omega = \omega + \epsilon \eta'$ is symplectic for small $\epsilon > 0$, and all F_i are symplectic with respect to ω' . \square

5.6. First step: a configuration of tori in \mathbb{T}^4 . Let $\mathbb{T}^4 = \mathbb{R}^4/\mathbb{Z}^4$, with coordinates x_1, \dots, x_4 . There are six embedded tori:

$$\begin{aligned} T_{12} &= \{(x_1, x_2, \alpha_3, \alpha_4)\} \subset \mathbb{T}^4, \quad T_{34} = \{(\alpha_1, \alpha_2, x_3, x_4)\} \subset \mathbb{T}^4, \\ T_{23} &= \{(\beta_1, x_2, x_3, \beta_4)\} \subset \mathbb{T}^4, \quad T_{14} = \{(x_1, \beta_2, \beta_3, x_4)\} \subset \mathbb{T}^4, \\ T_{13} &= \{(x_1, \gamma_2, x_3, \gamma_4)\} \subset \mathbb{T}^4, \quad T_{24} = \{(\gamma_1, x_2, \gamma_3, x_4)\} \subset \mathbb{T}^4, \end{aligned}$$

where $\alpha_i, \beta_i, \gamma_i$ are generic numbers which may be fixed when necessary. We get a configuration of six tori intersecting transversely in pairs $T_{12} \cap T_{34}$, $T_{23} \cap T_{14}$ and $T_{13} \cap T_{24}$, each pair intersects in a single point. The choice of the generic numbers ensures that one can have “parallel” disjoint copies T'_{ij} of T_{ij} .

Consider a symplectic form

$$\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + dx_2 \wedge dx_3 + \delta dx_1 \wedge dx_4 + dx_2 \wedge dx_4 - \delta dx_1 \wedge dx_3,$$

where $\delta > 0$ is small. Note that all T_{ij} are symplectic with respect to ω , where T_{13} is given the reversed orientation. From here we easily see that the following intersections are positive $[T_{12}] \cdot [T_{34}] > 0$, $[T_{13}] \cdot [T_{24}] > 0$, $[T_{14}] \cdot [T_{23}] > 0$.

Consider now the following specific collection of three disjoint 2-tori, which are symplectic in (\mathbb{T}^4, ω) ,

$$\begin{aligned} T_{12} &= \{(x_1, x_2, 0, 0)\} \subset \mathbb{T}^4, \\ T_{13} &= \{(x_1, 0, x_3, \frac{1}{2})\} \subset \mathbb{T}^4, \\ T_{14} &= \{(x_1, \frac{1}{2}, \frac{1}{2}, x_4)\} \subset \mathbb{T}^4 \end{aligned}$$

We shall do a Gompf connected sum along each of T_{12}, T_{13} and T_{14} . For this, we cut out tubular neighbourhoods of T_{12}, T_{13} and T_{14} of some small radius $\varepsilon > 0$.

$$\begin{aligned} Y &= \mathbb{T}^4 - (\nu(T_{12}) \cup \nu(T_{13}) \cup \nu(T_{14})) \\ &= \{(x_1, x_2, x_3, x_4) \mid |(x_3, x_4)| \geq \varepsilon, |(x_2, x_4 - \frac{1}{2})| \geq \varepsilon, |(x_2 - \frac{1}{2}, x_3 - \frac{1}{2})| \geq \varepsilon\}. \end{aligned}$$

We shall denote $\partial_{1j}Y = \partial\nu(T_{1j})$, $j = 2, 3, 4$, the three connected components of the boundary ∂Y .

Let us describe a configuration of certain Lagrangian tori and cylinders in Y to be used later.

$$\begin{aligned} C_1 &= \left\{ \left(x_1, -\delta\left(\frac{1}{2} - 2\varepsilon\right)(t - 1), 0, \varepsilon + \left(\frac{1}{2} - 2\varepsilon\right)t \right), t \in [0, 1] \right\}, \\ C_2 &= \left\{ \left(x_1, \frac{1}{2} + \delta\left(\frac{1}{2} - 2\varepsilon\right)(t - 1), \varepsilon + \left(\frac{1}{2} - 2\varepsilon\right)t, 0 \right), t \in [0, 1] \right\}, \\ T_1 &= \left\{ \left(\frac{1}{2} - \frac{\varepsilon}{\delta}(\sin \theta - \cos \theta), \varepsilon \cos \theta, x_3, \frac{1}{2} + \varepsilon \sin \theta \right), \theta \in [0, 2\pi] \right\}, \\ T_2 &= \left\{ \left(\frac{1}{2} - \frac{\varepsilon}{\delta}(\sin \theta + \cos \theta), \frac{1}{2} + \varepsilon \cos \theta, \frac{1}{2} + \varepsilon \sin \theta, x_4 \right), \theta \in [0, 2\pi] \right\}. \end{aligned}$$

Proposition 26. *If we choose δ and ε small enough, the cylinders C_1, C_2 and the tori T_1, T_2 satisfy the following:*

- (1) $C_1, C_2 \subset Y$, $T_1 \subset \partial_{13}Y$, $T_2 \subset \partial_{14}Y$,
- (2) $C_1 \cap C_2 = \emptyset$, $C_1 \cap T_2 = \emptyset$, $C_2 \cap T_1 = \emptyset$, $T_1 \cap T_2 = \emptyset$,
- (3) C_1 and T_1 intersect transversely in one point, and the same holds for C_2 and T_2 ,
- (4) C_1, C_2, T_1, T_2 are Lagrangian,
- (5) $\partial C_1 \subset \partial Y$ consists of two circles, one contained $\partial_{12}Y$ and another in $\partial_{13}Y$,
 $\partial C_2 \subset \partial Y$ consists of two circles, one contained $\partial_{12}Y$ and another in $\partial_{14}Y$.

Proof. The proof is obtained by a straightforward check up. It is easy to see that all of them are Lagrangian. For instance, for T_1 , the tangent space is generated by $-\frac{\varepsilon}{\delta}(\cos \theta + \sin \theta)\frac{\partial}{\partial x_1} - \varepsilon \sin \theta \frac{\partial}{\partial x_2} + \varepsilon \cos \theta \frac{\partial}{\partial x_4}$ and $\frac{\partial}{\partial x_3}$, and

$$\omega \left(-\frac{\varepsilon}{\delta}(\cos \theta + \sin \theta)\frac{\partial}{\partial x_1} - \varepsilon \sin \theta \frac{\partial}{\partial x_2} + \varepsilon \cos \theta \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_3} \right) = 0.$$

The torus $T_1 \subset \partial_{13}Y$ since its coordinates satisfy $|(x_2, x_4 - \frac{1}{2})| = \varepsilon$. Analogously $T_2 \subset \partial_{14}Y$.

Now, $\partial C_1 = \{(x_1, 0, 0, \varepsilon)\} \sqcup \{(x_1, -\delta(\frac{1}{2} - 2\varepsilon), 0, \frac{1}{2} - \varepsilon)\} \subset \partial_{12}Y \sqcup \partial_{13}Y$, and clearly $C_1 \subset Y$ since $x_3 = 0$ assures that it is well away from $\nu(T_{14})$. The statement for C_2 is similar.

It is clear that $C_1 \cap C_2 = \emptyset$. For $T_1 \cap T_2 = \emptyset$, it follows since they are in different boundary components. Also $C_1 \cap T_2 = \emptyset$ since $T_2 \subset \partial_{14}Y$ and $\partial C_1 \subset \partial_{12}Y \sqcup \partial_{13}Y$. Similarly, $C_2 \cap T_1 = \emptyset$. To compute $C_1 \cap T_1$, looking at the fourth coordinate we see that they coincide only for $t = 1, \theta = -\pi/2$ so there is only one point in $C_1 \cap T_1$. In the same way, $C_2 \cap T_2$ consists of one point (looking at the third coordinate). It is easy to check that these intersections are transverse. \square

5.7. Second step: The symplectic manifold Z . The normal bundles of $T_{1j} \subset \mathbb{T}^4$ are trivial. Therefore we can take three copies of the elliptic surface $E(1)$, call them $E(1)_2, E(1)_3$ and $E(1)_4$, with generic fibers F_2, F_3, F_4 , respectively, and form the Gompf symplectic sum

$$Z = \mathbb{T}^4 \#_{T_{12}=F_2} E(1)_2 \#_{T_{13}=F_3} E(1)_3 \#_{T_{14}=F_4} E(1)_4. \quad (5)$$

Using Lemma 24, we have that $\pi_1(Z)$ is isomorphic to the quotient of $\pi_1(\mathbb{T}^4)$ by the images of $\pi_1(T_{12}), \pi_1(T_{13}), \pi_1(T_{14})$, hence Z is simply-connected. Using the formula for the Euler characteristic of the Gompf symplectic sum in Subsection 5.3, one obtains

$$\chi(Z) = 36, b_2(Z) = 34.$$

Now we are going to construct 34 symplectic surfaces in Z . This will be done in several steps. First, let us focus on the first Gompf symplectic sum $\mathbb{T}^4 \#_{T_{12}=F_2} E(1)_2$. Call $E(1) = E(1)_2$, $F = F_{12}$, $T = T_{12}$. By Proposition 23 there are 9 sections E_1, \dots, E_9 of $E(1)$ which are spheres of self-intersection numbers (-1) intersecting F transversely at one point. By Lemma 22, we can glue them to (disjoint

parallel copies of) T_{34} , to get $S_1 = E_1 \# T_{34}, \dots, S_9 = E_9 \# T_{34}$, which are disjoint symplectic tori of self-intersection -1 . Now take a generic line $L \subset E(1)$ provided by Proposition 23, which intersects F in three points (and does not intersect any of the exceptional spheres E_i). This is a symplectic sphere which can be glued, by Lemma 22, to three parallel copies of T_{34} , to get a symplectic surface $S_{10} = L \# 3T_{34}$ of genus 3 and self-intersection 1, which is moreover disjoint from all the previous ones.

When doing the second and third Gompf symplectic sums in (5), we construct similar collections $S_{11}, \dots, S_{19}, S_{20}$ and $S_{21}, \dots, S_{29}, S_{30}$ of symplectic surfaces in Z , so that

- $g(S_1) = \dots = g(S_9) = 1, g(S_{11}) = \dots = g(S_{19}) = 1, g(S_{21}) = \dots = g(S_{29}) = 1, g(S_{10}) = g(S_{20}) = g(S_{30}) = 3.$
- $S_k \cdot S_k = -1, 1 \leq k \leq 9, S_{10} \cdot S_{10} = 1, S_{10+k} \cdot S_{10+k} = -1, 1 \leq k \leq 9, S_{20} \cdot S_{20} = 1, S_{20+k} \cdot S_{20+k} = -1, 1 \leq k \leq 9, S_{30} \cdot S_{30} = 1.$

All of them are disjoint since for constructing $S_{10+k}, k = 1, \dots, 10$, we glue with parallel copies of T_{24} , and for constructing $S_{20+k}, k = 1, \dots, 10$, we glue with parallel copies of T_{23} . We can arrange as many copies as we wish of T_{34}, T_{24}, T_{23} which do not intersect.

The four remaining surfaces are constructed as follows. Consider the (Lagrangian) cylinders C_1, C_2 and tori T_1, T_2 from Proposition 26. Recall that they are contained in Y , so they are disjoint with the tori $T_{1j}, j = 2, 3, 4$. Moreover, we can take collections of parallel copies of T_{34}, T_{24}, T_{23} which do not intersect any of C_1, C_2, T_1, T_2 . Therefore we can assume that C_i and T_i are disjoint from S_1, \dots, S_{30} in Z .

We use the cylinder C_i to construct Lagrangian spheres in Z as follows. The boundary of ∂C_1 in $\partial_{12}Y$ is a circle γ . We arrange the identification $\partial(E(1)_2 - \nu(F_2)) = F_2 \times S^1 \cong \partial_{12}Y = T_{12} \times S^1$ to match this circle with a vanishing cycle of the elliptic fibration $E(1)_2$ (see Subsection 5.4). Let V be the vanishing disk in $E(1)_2$, which is a Lagrangian (-1) -disk. This can be glued to C_1 to obtain a Lagrangian submanifold $V \cup_\gamma C_1$ of self-intersection -1 . To make the gluing smooth, we may need to change the gluing in the Gompf connected sum as follows: the gluing region is a neighbourhood of $Y = F \times S^1$ of the form $F \times S^1 \times (-\epsilon, \epsilon)$, where the symplectic form is $\omega_F + d\theta \wedge dt$, and the Lagrangian has tangent space at s spanned by γ' and a vector $a \frac{\partial}{\partial \theta} + b \frac{\partial}{\partial t}$. A diffeomorphism of the form $(\theta, s) \mapsto (\theta + g(s), s)$ can serve to arrange $a = 0$, so that the Lagrangian enters the gluing region in the radial direction and thus can be glued without corner. Finally, gluing the other boundary component of ∂C_1 with a vanishing disk in $E(1)_3$, we get a Lagrangian (-2) -sphere L_1 . This intersects T_1 transversely at one point.

In a similar way we obtain another pair L_2, T_2 of a Lagrangian (-2) -sphere and Lagrangian torus of self-intersection 0, both intersecting transversely at one point. We can arrange that L_1, L_2 are disjoint, because by Proposition 23 we can choose two different vanishing cycles (hence disjoint) in $E(1)_2$, to match the two boundary components of C_1, C_2 in $\partial_{12}Y$, which are homologous cycles.

Looking at the intersection form, we see that the 34 surfaces S_1, \dots, S_{30} and L_1, L_2, T_1, T_2 are independent in homology, hence they span $H_2(Z, \mathbb{Q})$. Finally, we apply Lemma 25 to change slightly the symplectic form so that all these Lagrangian surfaces become symplectic. Moreover, the proof of Lemma 25 shows that we can deform the symplectic form so that both pairs (L_1, T_1) and (L_2, T_2) intersect positively, so we assume this.

5.8. Making all symplectic surfaces disjoint. To make the surfaces in Z disjoint we have to do the following process with both pairs L_1, T_1 and L_2, T_2 . Let L, T a pair of a symplectic sphere and a symplectic torus with $L \cdot L = -2, L \cdot T = 1, T \cdot T = 0$. Take a parallel copy of T , call it T' , displacing via the normal bundle. Resolve the intersection point $T' \cap L$ with the process of Subsection 5.1 to get a torus T'' homologous to $T' + L$. Hence $T'' \cdot T'' = (T + L)^2 = 0$ and $T'' \cdot T = (T + L) \cdot T = 1$. Therefore T and T'' intersect at one point, say p . Locally, the model around p is determined by the equation $z \cdot w = 0$, where $T = \{z = 0\}$, and $T'' = \{w = 0\}$. Consider $T + T''$ and resolve the singularity producing a symplectic genus 2 surface Σ . We move it to intersect T and T'' in the same point p . Locally, it is the same as to write down the equation $(z - \varepsilon) \cdot (w - \varepsilon) = \varepsilon^2$. The equalities

$$\Sigma \cdot T = (T + T'') \cdot T = 1, \Sigma \cdot T'' = (T + T'') \cdot T'' = 1,$$

show that p is the only intersection point of the three surfaces T, T'', Σ , and that they intersect transversely. Moreover, $\Sigma^2 = (T + T'')^2 = 2$. Blowing up at p we get a symplectic manifold $\tilde{Z} = Z \# \overline{\mathbb{C}P^2}$, where the proper transforms $\tilde{T}, \tilde{T}'', \tilde{\Sigma}$ are disjoint symplectic surfaces of genus 1, 1, 2 and self-intersection numbers $-1, -1, 1$ (see Subsection 5.2). They generate the same 3-dimensional space in homology, as T, T'' and the exceptional sphere $E \in H_2(\tilde{Z}, \mathbb{Z})$.

Using this method for both pairs L_1, T_1 and L_2, T_2 , we end up with the symplectic manifold $X = Z \# 2\overline{\mathbb{C}P^2}$, with $b_2(X) = 36$, and with 36 disjoint symplectic surfaces $S_1, \dots, S_{30}, \tilde{T}_1, \tilde{T}_1'', \tilde{\Sigma}_1, \tilde{T}_2, \tilde{T}_2'', \tilde{\Sigma}_2$. This forces that these 36 surfaces generate the homology of X . The genus and self-intersections of the surfaces are those stated in Theorem 21. This finishes the proof.

Corollary 27. *Take a prime p , and $g_i = g(S_i)$ as given in Theorem 21. Then there is a 5-dimensional K-contact manifold M with $H_1(M, \mathbb{Z}) = 0$ and*

$$H_2(M, \mathbb{Z}) = \mathbb{Z}^{35} \oplus \bigoplus_{i=1}^{36} (\mathbb{Z}/p^i)^{2g_i}.$$

Proof. Consider the symplectic manifold (X, ω) provided by Theorem 21, and let S_i , $1 \leq i \leq 36$, be the collection of disjoint symplectic surfaces. Put coefficients $m_i = p^i$ for S_i . Using Proposition 7, we give X the structure of a symplectic orbifold with isotropy surfaces S_i of multiplicities m_i . By Lemma 9, X admits an almost Kähler orbifold structure. Lemma 19 implies that there exists a Seifert bundle $M \rightarrow X$ such that $c_1(M/X) = [\omega]$, and by Theorem 20, M admits a K-contact structure.

We compute the homology of M using Theorem 16. As X is simply connected, $H_1(X, \mathbb{Z}) = 0$. By Lemma 19, we can arrange that $c_1(M/\mu) \in H^2(X, \mathbb{Z})$ is primitive. Now $k+1 = b_2(X) = 36$, so $H^2(X, \mathbb{Z}) = \mathbb{Z}^{36}$. The map $H^2(X, \mathbb{Z}) \rightarrow H^2(S_i, \mathbb{Z})$ sends $[S_j]$ to zero, $j \neq i$, since all S_i are disjoint. It sends $[S_i]$ to S_i^2 , hence $H^2(X, \mathbb{Z}) \rightarrow H^2(S_i, \mathbb{Z}/p^{e_i})$ sends $[S_i]$ to $S_i^2 \pmod{p^i}$. Given the self-intersection numbers in Theorem 21, this is non-zero. So

$$H^2(X, \mathbb{Z}) \rightarrow \sum H^2(S_i, \mathbb{Z}/p^i)$$

is surjective. Hence $H_1(M, \mathbb{Z}) = 0$. The result follows. \square

6. KÄHLER SURFACES WITH MANY DISJOINT COMPLEX CURVES

Now we want to find obstructions for the existence of Sasakian 5-dimensional manifolds. In particular, we aim to prove that the 5-manifold constructed in the previous section, which admits a K-contact structure, cannot admit a Sasakian structure.

The proof of Theorem 1 follows from Corollary 27 and the following:

Proposition 28. *Let M be a 5-dimensional manifold with $H_1(M, \mathbb{Z}) = 0$ and*

$$H_2(M, \mathbb{Z}) = \mathbb{Z}^{35} \oplus \bigoplus_{i=1}^{36} (\mathbb{Z}/p^i)^{2g_i}.$$

where $g_i = g(S_i)$ are the numbers given in Theorem 21, and p is a prime number. Then M does not admit a semi-regular Sasakian structure.

Proof. Let M be a 5-dimensional manifold with $H_1(M, \mathbb{Z}) = 0$ which admits a Sasakian structure. Then it also admits a quasi-regular Sasakian structure. This means that M is a Seifert bundle over a Kähler orbifold $\pi : M \rightarrow X$, by Theorem 18. By Corollary 17, $H_1(X, \mathbb{Z}) = 0$, $H_2(X, \mathbb{Z}) = \mathbb{Z}^{36}$ and the ramification locus contains a collection of 36 disjoint surfaces D_i with $g(D_i) = g_i$.

If the Sasakian structure is semi-regular, then X is a smooth Kähler manifold. By Proposition 10, the ramification locus consists of smooth Kähler curves. We see in Theorem 29 below that this is not possible. \square

A smooth Kähler manifold with disjoint complex curves spanning its homology is a rare phenomenon. We have the following first result in this direction.

Theorem 29. *Let S be a smooth Kähler surface with $H_1(S, \mathbb{Q}) = 0$ and containing D_1, \dots, D_b , $b = b_2(S)$, smooth disjoint complex curves with $g(D_i) = g_i > 0$. Assume that:*

- at least two g_i are bigger than 1,
- $g = \max\{g_i\} \leq 3$.

Then $b \leq 2g + 3$.

Proof. First, it is clear that $[D_1], \dots, [D_b]$ are a basis of $H^2(S, \mathbb{Q})$. As these are classes of type $(1, 1)$, we have that $h^{1,1} = b$ and geometric genus $p_g = h^{2,0} = 0$. The irregularity is $q = h^{1,0} = 0$ since $b_1 = 0$. Therefore Noether's formula [1] says that

$$\frac{1}{12}(K_S^2 + c_2(S)) = \chi(\mathcal{O}_S) = 1 - q + p_g = 1.$$

Note that $c_2(S) = \chi(S) = 2 + b$, since $b = b_2$ and $b_1 = b_3 = 0$. Therefore $K_S^2 = 10 - b$, where K_S is the canonical divisor of S .

By the Riemann-Hodge relations, the signature of $H^{1,1}(S)$ is $(1, b-1)$. Therefore, we can suppose $D_1^2 = m_1$, $D_i^2 = -m_i$, $i = 2, \dots, b$, where all m_i are positive integer numbers. By the adjunction equality, we have

$$K_S \cdot D_i + D_i^2 = 2g_i - 2,$$

so $K_S \cdot D_i = 2g_i - 2 - D_i^2$, and hence

$$K_S = \sum_{i=1}^b \frac{2g_i - 2 - D_i^2}{D_i^2} D_i$$

and

$$K_S^2 = \sum_{i=1}^b \frac{(2g_i - 2 - D_i^2)^2}{D_i^2}.$$

For $i \geq 2$, we have

$$\frac{(2g_i - 2 - D_i^2)^2}{D_i^2} = -\frac{(2g_i - 2 + m_i)}{m_i} (2g_i - 2 + m_i) \leq -(2g_i - 2 + m_i) \leq -1,$$

since $2g_i - 2 \geq 0$. Then

$$10 - b = K_S^2 \leq \frac{(2g_1 - 2 - D_1^2)^2}{D_1^2} - (b - 1).$$

With the hypothesis that at least one g_i , $i \geq 2$, satisfies that $g_i > 1$, we have an strict inequality. So

$$\frac{(2g_1 - 2 - m_1)^2}{m_1} \geq 10.$$

This is rewritten as $m_1^2 - (4g_1 + 6)m_1 + 4(g_1 - 1)^2 \geq 0$. Hence

$$m_1 \geq 2g_1 + 3 + \sqrt{20g_1 + 5} \quad \text{or} \quad m_1 \leq 2g_1 + 3 - \sqrt{20g_1 + 5}.$$

For $g_1 \leq 3$, we have that the second inequality is impossible (since $m_1 \geq 1$). Hence $m_1 \geq 2g_1 + 3$.

Now we have that there is a curve D_1 of genus g_1 with self-intersection $D_1^2 \geq 2g_1 - 1$. Take the line bundle $L = \mathcal{O}(D_1)$. This has $m_1 = \deg(L|_{D_1}) \geq 2g_1 - 1$, so $L|_{D_1}$ is very ample. In particular, there is a section $s \in H^0(L|_{D_1})$ vanishing exactly at m_1 distinct points $Z \subset D_1$. The long exact sequence in cohomology associated to $0 \rightarrow \mathcal{O} \rightarrow L \rightarrow L|_{D_1} \rightarrow 0$, together with the fact that $H^1(\mathcal{O}) = H^{0,1}(S) = 0$, gives an exact sequence

$$0 \rightarrow \mathbb{C} \rightarrow H^0(L) \rightarrow H^0(L|_{D_1}) \rightarrow 0,$$

Take the preimage of the section $s \in H^0(L|_{D_1})$. This is a 2-dimensional subspace of $H^0(L)$. It gives a Lefschetz pencil $\mathbb{P}^1 \subset \mathbb{P}(H^0(L))$ of sections whose zero sets are curves going through Z . Blow-up Z to get a smooth complex surface \tilde{S} and a Lefschetz fibration

$$\pi : \tilde{S} \longrightarrow \mathbb{P}^1$$

with the proper transform of D_1 , say $C_1 = \tilde{D}_1$ as one smooth fiber of genus g_1 . The other D_i , $2 \leq i \leq b$, are not touched by the blow-up loci, so we do not change their name.

Now let E_j , $j = 1, \dots, m_i$, be the exceptional divisors of the blow-up map $\tilde{S} \rightarrow S$. These are sections of π . Note that $C_1, E_1, \dots, E_{m_1}, D_2, \dots, D_b$ are a basis of $H_2(\tilde{S}, \mathbb{Q})$. Since $D_j \cdot E_1 = 0$, we have that D_j is contained in a fiber for all $i = 2, \dots, b$. Note that it follows that $g_i \leq g_1$, for $i \geq 2$, since the genus of a component of a singular fiber cannot be bigger than the genus of the generic fiber. So $g = g_1$.

Let us see that π is a relatively minimal fibration. This means that there are no (-1) -rational curves contained in a fiber. Suppose that B is such a curve. If B intersects a section say E_1 , then $B + E_1$ is a rational nodal curve of self-intersection zero. This implies that there is a linear system of rational curves of self-intersection zero and hence \tilde{S} is ruled. However, $(B + E_1) \cdot D_3 = 0$, and D_3 has positive genus. This is not possible (a curve of positive genus survives in a minimal model of \tilde{S} , hence it should be intersected by the ruling). Suppose that B does not intersect any section. Then B is contained in a fiber. If B does not intersect any D_j then it is homologically trivial. Suppose it intersect some D_k in some fiber F . Let F_1, \dots, F_k be the irreducible components of F . By [1, (III.8.2)], the span of $\langle F_1, \dots, F_k \rangle$ has dimension k , and subject to the only relation $C_1 = F = \sum a_i F_i$, for some a_i . Removing the components that do intersect the exceptional divisors, the rest of the components, together with the D_i and the E_j , should be independent. Therefore there cannot be more components not intersecting the E_j than those provided by the D_k in the fiber, hence such B does not appear.

Now, as in [23] and [10], write

$$\begin{aligned} K_{\tilde{S}/\mathbb{P}^1}^2 &= K_{\tilde{S}}^2 - 8(g-1)(-1) = 10 - b - m_1 + 8g - 8, \\ \chi_\pi &= \chi(\mathcal{O}_{\tilde{S}}) - (g-1)(-1) = 1 + g - 1 = g, \\ \lambda_\pi &= K_{\tilde{S}/\mathbb{P}^1}^2 / \chi_\pi = (2 - b - m_1 + 8g) / g. \end{aligned}$$

By [23], for any relatively minimal fibration of genus $g \geq 2$, we have $4 - 4/g \leq \lambda_\pi \leq 12$. The first inequality implies that $4g - 4 \leq 2 - b - m_1 + 8g \leq 2 - b - (2g + 3) + 8g$ hence $b \leq 2g + 3$. \square

Remark 30. *The proof of Theorem 29 also works when we have all complex curves of genus $g_i = 1$. We only have to note that automatically $m_1 \geq 1$, and this is enough to construct a Lefschetz fibration.*

To extend the arguments of this paper to quasi-regular Sasakian manifolds (and hence to all Sasakian manifolds), we need a version of Theorem 29 that covers

the case that S is a cyclic Kähler orbifold. The argument should run as follows: desingularize each orbifold point (this is a Hirzebruch-Jung desingularisation [1]), creating a tree of rational curves of negative self-intersection, and bound $K_{\tilde{S}}^2$ for the desingularisation $\tilde{S} \rightarrow S$. The authors have only managed to make this argument work for the case where all complex curves are of genus $g_i = 1$. Unfortunately, we have not been able to construct a symplectic manifold X with $H_1(X, \mathbb{Z}) = 0$ and $b = b_2(X)$ disjoint symplectic tori in X .

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FACULTAD DE CIENCIAS MATEMÁTICAS, UNIVERSIDAD COMPLUTENSE DE MADRID, PLAZA DE CIENCIAS 3, 28040 MADRID, SPAIN

INSTITUTO DE CIENCIAS MATEMÁTICAS (CSIC-UAM-UC3M-UCM), C/ NICOLÁS CABRERA 15, 28049 MADRID, SPAIN

E-mail address: `vicente.munoz@mat.ucm.es`

FACULTAD DE CIENCIAS MATEMÁTICAS, UNIVERSIDAD COMPLUTENSE DE MADRID, PLAZA DE CIENCIAS 3, 28040 MADRID, SPAIN

INSTITUTO DE CIENCIAS MATEMÁTICAS (CSIC-UAM-UC3M-UCM), C/ NICOLÁS CABRERA 15, 28049 MADRID, SPAIN

E-mail address: `jarcarulli@ucm.es`

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF WARMIA AND MAZURY, SŁONECZNA 54, 10-710, OLSZTYN, POLAND

E-mail address: `tralle@matman.uwm.edu.pl`